2018 6th Iranian Joint Congress on Fuzzy and Intelligent Systems (CFIS)

# A Metric on $[0, 1]$ Which Makes it A Topological SL-Algebra

1st Seyed Mohammad Amin Khatami
*Department of Computer Science*
*Birjand University of Technology*
Birjand, Iran
khatami@birjandut.ac.ir

***Abstract*—We introduce a metric on $[0, 1]$ as a dual of a BL-algebra, i.e. SL-algebra, which makes all lattice operations continuous. As an application of this result, one could use the ultraproduct method on residuated lattice and their dual.**

***Index Terms*—SL-algebra, BL-algebra, continuous t-conorm, continuous t-norm, metric**

## I. Introduction

The beginning of BL-algebras goes back to the study of continuous t-norm based fuzzy logics by Hajek [4]. A more general algebraic structure originated in logics without contractions is residuated lattice. The oldest such structure which is appeared in classical logic is Boolean algebra. All of this structures are studied from the algebraic and topological point of view. Bozooei et.al in [2], [6] introduce the notion of topological BL-algebras and then in [1] they study the metrizability on BL-algebras. Here, we introduce a metric on the dual of the standard BL-algebra $[0, 1]$, called SL-algebra. Using this metric space, one could get a topology on the standard BL-algebra $[0, 1]$ which makes it a topological BL-algebra.

## II. Preliminaries

In this section we have an outline about the standard BL-algebra $[0, 1]$ and its dual.

**Definition 1.** [4] A BL-algebra is an algebra

$$\mathcal{L} = (L, \wedge, \vee, *, \rightarrow, 0, 1)$$

of type $(2, 2, 2, 2, 0, 0)$ satisfies the following conditions:

- $(L, \wedge, \vee, 0, 1)$ is a bounded lattice with greatest element 1 and smallest element 0,
- $(L, *, 1)$ is an Abelian monoid,
- $\rightarrow$ is the residua of $*$, i.e.,

$$a \leq b \rightarrow c \text{ iff } a * b \leq c \text{ for all } a, b, c \in L,$$

- $a \wedge b = a * (a \rightarrow b)$ for all $a, b \in L$,
- $(a \rightarrow b) \vee (b \rightarrow a)$ for all $a, b \in L$.

978-1-5386-2836-2/18/$31.00 ©2018 IEEE

The BL-algebras on the real segment $[0, 1]$ which is defined by a continuous t-norms, are called t-algebras. A continuous t-norm is a continuous function $T$ (in Euclidean topology) that is commutative, associative, non-decreasing on both arguments, and $T(1, a) = a$ for all $a \in [0, 1]$. However, [4] and [3] mentioned that when $[0, 1]$ endowed with the set of operators $\{\wedge, \vee, *, \rightarrow\}$ to be a BL-algebra, then $*$ becomes a continuous t-norm.

The dual notion of t-norm is t-conorm or s-norm. A continuous s-norm is a continuous function $S : [0, 1]^2 \rightarrow [0, 1]$ (in Euclidean topology) that is commutative, associative, non-decreasing on both arguments, and $S(0, a) = a$ for all $a \in [0, 1]$. The residua of the continuous s-norm $S$ is the unique operator $R_S : [0, 1]^2 \rightarrow [0, 1]$, defined by the adjoint property,

$$\text{for all } a, b, c \in [0, 1], \ S(a, b) \geq z \text{ iff } a \geq R_S(b, c).$$

Continuity of $S$ implies that $R_S(a, b) = \min\{c : S(c, a) \geq b\}$. Among well-known continuous s-norms and their residua, one could refer to the Łukasiewicz and Gödel s-norm.

$$S_L(x, y) = \min\{x + y, 1\} \ , \ R_L(x, y) = \begin{cases} 0 & x \geq y \\ y - x & x < y \end{cases}$$

$$S_G(x, y) = \max\{x, y\} \ , \ R_G(x, y) = \begin{cases} 0 & x \geq y \\ y & x < y \end{cases}$$

Furthermore, an easy argument show that

$$S\big(a, R_S(a, b)\big) = \max\{a, b\}$$

for all $a, b \in L$. So the dual notion of BL-algebra could be considered as follows.

**Definition 2.** [4] An SL-algebra is an algebra

$$\mathcal{L} = (L, \curlywedge, \curlyvee, \star, \dot{\rightarrow}, 0, 1)$$

of type $(2, 2, 2, 2, 0, 0)$ satisfies the following conditions:

- $(L, \curlywedge, \curlyvee, 0, 1)$ is a bounded lattice with greatest element 1 and smallest element 0
(note that $a \leq b$ means $a \curlyvee b = a$, $\curlywedge$ is the standard join operator $\vee$ that is

$$a \curlywedge b = a \vee b = \sup\{a, b\},$$

and $\dot{\vee}$ is the standard meet operator $\wedge$ that is
$$a \dot{\vee} b = a \wedge b = \inf\{a, b\}),$$

- $(L, \star, 0)$ is an Abelian monoid,
- $\dot{\rightarrow}$ is the residua of $\star$, i.e.,
$$a \geq b \dot{\rightarrow} c \text{ iff } a \star b \geq c \text{ for all } a, b, c \in L,$$
- $a \dot{\wedge} b = a \star (a \dot{\rightarrow} b)$ for all $a, b \in L$,
- $(a \dot{\rightarrow} b) \dot{\vee} (b \dot{\rightarrow} a)$ for all $a, b \in L$.

Any SL-algebra is denoted by it's domain. Like the BL-algebras, when $[0, 1]$ endowed with the set of operators
$$\{\dot{\wedge}, \dot{\vee}, \star, \dot{\rightarrow}\}$$
to be an SL-algebra, then $\star$ becomes a continuous s-norm.

**Proposition 1.** *[4] Let L be an SL-algebra. then the following properties hold:*

1) $a \star (a \dot{\rightarrow} b) = a \dot{\wedge} b = \sup\{a, b\}$,
2) $a \dot{\rightarrow} b \geq (b \dot{\rightarrow} c) \dot{\rightarrow} (a \dot{\rightarrow} c)$,
3) $a \star b \geq \sup\{a, b\}$,
4) $b \geq a \dot{\rightarrow} b$,
5) *if* $a \geq b$ *then* $a \dot{\rightarrow} b = 0$,
6) $(a \dot{\rightarrow} b) \star (a' \dot{\rightarrow} b') \geq (a \star a') \dot{\rightarrow} (b \star b')$

## III. Main results

**Theorem 1.** *Let $\star$ be a continuous s-norm weaker than the Łukasiewicz s-norm, i.e. $a \star b \leq S_L(a, b)$ for all $a, b \in [0, 1]$. Furthermore, let $\dot{\rightarrow}$ be the residue of $\star$. Then*
$$d_\star(a, b) = \begin{cases} a \dot{\rightarrow} b & a \leq b \\ b \dot{\rightarrow} a & a > b \end{cases}$$
*is a metric on* $[0, 1]$.

*Proof.* Clearly, $d_\star(a, a) = 0$ for $a \in [0, 1]$. Furthermore, if $d_\star(a, b) = 0$ and we assume that $a < b$, then
$$0 = d_\star(a, b) = a \dot{\rightarrow} b = min\{c : c \star a \geq b\}.$$
Thus, $0 \in \{c : c \star a \geq b\}$ which means that $a \geq b$, a contradiction.

Symmetric property of $d_\star$ is clear. To prove the triangle inequality, by Proposition 1(2) for arbitrary $z, b, c, \in [0, 1]$ we have $a \dot{\rightarrow} c \geq (c \dot{\rightarrow} b) \dot{\rightarrow} (a \dot{\rightarrow} b)$ which near the adjointness of $\star$ and $\dot{\rightarrow}$ implies that
$$(a \dot{\rightarrow} c) \star (c \dot{\rightarrow} b) \geq a \dot{\rightarrow} b.$$
But, $\star$ is weaker than $S_L$. So
$$\begin{aligned} a \dot{\rightarrow} b &\leq (a \dot{\rightarrow} c) \star (c \dot{\rightarrow} b) \\ &\leq S_L (a \dot{\rightarrow} c, c \dot{\rightarrow} b) \\ &\leq d_\star(a, c) + d_\star(c, b). \end{aligned}$$
A similar argument show that $b \dot{\rightarrow} a \leq d_\star(a, c) + d_\star(c, b)$, that completes the proof. □

The correspnding metric for Gödel s-norm and Łukasiewicz s-norm are as follows
$$d_L(x, y) = |x - y| \ , \ d_G(x, y) = \begin{cases} \max\{x, y\} & x \neq y \\ 0 & x = y \end{cases}$$

The operations of the Łukasiewicz SL-algebra $[0, 1]$ are continuous with respect to the Euclidean topology $d_L(x, y) = |x-y|$. However, continuity of operations does not hold in general, as for example $\dot{\rightarrow}_G$ is not continuous. We show that for any continuous s-norm $\star$ and it's residua, all of the operations of the corresponding SL-algebra $([0, 1], \wedge, \vee, \star, \dot{\rightarrow}, 0, 1)$ are continuous with respect to induced topology of the metric $d_\star$ introduced in Theorem 1. Note that the Euclidean topology on $[0, 1]^2$ is usually defined by
$$d\big((x_1, x_2), (y_1, y_2)\big) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$
which is equivalent to the maximum metric
$$\max\{|x_1 - y_1|, |x_2 - y_2|\}$$
and taxicab metric
$$|x_1 - y_1| + |x_2 - y_2|.$$
Here we use a metric, similar to the taxicab metric on $[0, 1]^2$ which is made by the metric $d_\star$. Indeed, for any continuous s-norm $\star$ weaker than the Łukasiewicz s-norm, the mapping
$$\mathbf{d}_\star\big((x_1, x_2), (y_1, y_2)\big) = d_\star(x_1, y_1) \star d_\star(x_2, y_2)$$
define a metric on $[0, 1]^2$. We only check the transitivity of $\mathbf{d}$. In fact by Proposition 1(3) together with the transitivity of metric function $d_\star$, and basic properties of s-norm $\star$ we have
$$\begin{aligned} & \mathbf{d}_\star\big((x_1, x_2), (y_1, y_2)\big) \\ = \ & d_\star(x_1, y_1) \star d_\star(x_2, y_2) \\ \leq \ & \big(d_\star(x_1, z_1) \star d_\star(z_1, y_1)\big) \star \big(d_\star(x_2, z_2) \star d_\star(z_2, y_2)\big) \\ = \ & \big(d_\star(x_1, z_1) \star d_\star(x_2, z_2)\big) \star \big((d_\star(z_1, y_1) \star d_\star(z_2, y_2)\big) \\ = \ & \mathbf{d}_\star\big((x_1, x_2), (z_1, z_2)\big) \star \mathbf{d}_\star\big((z_1, z_2), (y_1, y_2)\big) \\ \leq \ & S_L\Big(\mathbf{d}_\star\big((x_1, x_2), (z_1, z_2)\big), \mathbf{d}_\star\big((z_1, z_2), (y_1, y_2)\big)\Big) \\ \leq \ & \mathbf{d}_\star\big((x_1, x_2), (z_1, z_2)\big) + \mathbf{d}_\star\big((z_1, z_2), (y_1, y_2)\big) \end{aligned}$$
Finally, we show that the all of the operations of the SL-algebra $[0, 1]$ are continuous functions with respect to the induces topology from $d_\star$ and $\mathbf{d}_\star$ on $[0, 1]$ and $[0, 1]^2$. As we know that
$$a \dot{\wedge} b = \sup\{a, b\} = a \star (a \dot{\rightarrow} b),$$
$$a \dot{\vee} b = \inf\{a, b\} = \big((a \dot{\rightarrow} b) \dot{\rightarrow} b\big) \vee \big((b \dot{\rightarrow} a) \dot{\rightarrow} a\big),$$
we only must prove the assertion for $\star$ and $\dot{\rightarrow}$.

**Theorem 2.** *Let $\star$ and $\dot{\rightarrow}$ be as in Theorem 1. then*
$$\star : ([0, 1]^2, \mathbf{d}_\star) \to ([0, 1], d_\star)$$
*and*
$$\dot{\rightarrow}: ([0, 1]^2, \mathbf{d}_\star) \to ([0, 1], d_\star)$$
*are continuous functions.*

*Proof.* By using Proposition 1(6) we have
$$\begin{aligned} (a_1 \star a_2) \dot{\rightarrow} (b_1 \star b_2) &\leq (a_1 \dot{\rightarrow} b_1) \star (a_2 \dot{\rightarrow} b_2) \\ &\leq d_\star(a_1, b_1) \star d_\star(a_2, b_2) \\ &= \mathbf{d}_\star\big((a_1, a_2), (b_1, b_2)\big), \end{aligned}$$

which shows that $\star$ is a uniformly continuous function. For continuity of $\dot{\to}$, by Proposition 1(2) we have

$$a_1 \dot{\to} b_1 \geq (b_1 \dot{\to} b_2) \dot{\to} (a_1 \dot{\to} b_2).$$

Now, by adjointness we get

$$(a_1 \dot{\to} b_1) \star (b_1 \dot{\to} b_2) \geq a_1 \dot{\to} b_2.$$

Again using Proposition 1(2) we have

$$a_1 \dot{\to} b_2 \geq (b_2 \dot{\to} a_2) \dot{\to} (a_1 \dot{\to} a_2)$$

which beside the above inequality leads to

$$(a_1 \dot{\to} b_1) \star (b_1 \dot{\to} b_2) \geq (b_2 \dot{\to} a_2) \dot{\to} (a_1 \dot{\to} a_2).$$

Again, by adjointness

$$\big((a_1 \dot{\to} b_1) \star (b_1 \dot{\to} b_2)\big) \star (b_2 \dot{\to} a_2) \geq (a_1 \dot{\to} a_2)$$

and by properties of s-norm

$$\big((a_1 \dot{\to} b_1) \star (b_2 \dot{\to} a_2)\big) \star (b_1 \dot{\to} b_2) \geq (a_1 \dot{\to} a_2).$$

Finally, again by adjointness

$$\begin{aligned}(b_1 \dot{\to} b_2) \dot{\to} (a_1 \dot{\to} a_2) &\leq (a_1 \dot{\to} b_1) \star (b_2 \dot{\to} a_2)\\ &\leq d_\star(a_1, b_1) \star d_\star(a_2, b_2)\\ &= \mathbf{d}_\star\big((a_1, a_2), (b_1, b_2)\big),\end{aligned}$$

which is done the proof. □

## Conclusion

Let $L = ([0,1], \wedge, \vee, *, \to, 0, 1)$ be a BL-algebra. Furthermore assume that $*$ be a continuous t-norm stronger than the Łukasiewicz t-norm and $x \star y = (1-x) * (1-y)$. Then $\star$ is a continuous s-norm weaker than the Łukasiewicz s-norm. The dual of the induced topology by the metric $d_\star$, makes $L$ a topological BL-algebra. We guess that for any continuous t-norm (not only necessarily stronger than the Łukasiewicz t-norm) the issue will be correct, that is we get a topological BL-algebra. This topology could be used in the ultraproduct method in t-norm based fuzzy logics. We use this fact to prove the $K$-compactness theorem for Gödel logic [5].

## References


[1] R. A. Borzooei, G. R. Rezaei, and N. Kouhestani, "Metrizability on (semi)topological BL-algebras," Soft Computing, vol. 16, pp. 1681–1690, October 2012.

[2] R. A. Borzooei, G. R. Rezaei, and N. Kouhestani, "On (semi) topological BL-algebras," Irainian journal of mathematical science and informatics, vol. 6, pp. 59–77, May 2011.

[3] R. Cignoli, F. Esteva, and L. Godo, "Basic Fuzzy Logic is the logic of continuous t-norms and their residua," Soft Computing, vol. 4, pp. 106–112, July 2000.

[4] P. Hájek, Metamathematics of fuzzy logic, Springer, Berlin: Heidelberg, 1998.

[5] S. M. A. Khatami and M. Pourmahdian, "On the compactness property of extensions of first-order Gödel logic," Iranian Journal of Fuzzy Systems, vol. 12, pp. 101–121, July and August 2015.

[6] O. Zahiri, R. A. Borzooei, "Topology on BL-algebras," Fuzzy Sets and Systems, vol. 289, pp. 137–150, April 2016.