A-10-872-2

# A many-valued logic whose corresponding algebra is an ordered Abelian group

Seyed Mohammad Amin Khatami[1]
Department of Computer Science, Birjand University of Technology, Birjand, Iran

**Abstract**

We study an extension of Gödel propositional logic whose corresponding algebra is an ordered Abelian group. Then we expand the ideas to first-order case of this logic.

**Keywords:** Gödel logic, Łukasiewicz logic, totally ordered Abelian group
**Mathematics Subject Classification [2010]:** 03B50, 03B52, 03G25

## 1 Introduction

There is a direct relationship between the proof system of any propositional logic and the class of algebraic structures which verify exactly the provable formulae of that logic. For example Boolean algebras are the algebraic counterpart of classical propositional logic and Heyting algebras correspond to intuitionistic propositional logic.

In many-valued logic, specially after introducing infinite valued Łukasiewicz logic and then fuzzy logics, the numbers of $[0, 1]$ are considered as the possible standard truth-values which the logical sentences can be assigned to. However, any lattice structure could be considered as the truth value set. The structure of the desirable truth value set of a logic, commonly formed by its application or by its proof system. For continuous t-norm based many-valued logic, a general framework based on proof system is introduced by Hajek [1]. Indeed, any propositional logic based on continuous t-norm, whose primary connectives are $\{\&, \to, \bot\}$ in which & interpret by a continuous t-norm $T : [0, 1]^2 \to [0, 1]$, $\to$ interpreted by residuum of $*$ which is defined by $x \Rightarrow y = \sup\{z : T(z, x) \le y\}$, and $\bot$ interpreted by 0, could be axiomatized by the axioms and rules listed in Table 1. The class of all equivalent formulae, $\mathbf{L}$ with the order $\le$, defined by

Table 1: Axioms and Rules of BL

| Axioms of Basic Logic |
|---|
| (A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$ |
| (A2) $(\varphi\&\psi) \to \varphi$ |
| (A3) $(\varphi\&\psi) \to (\psi\&\varphi)$ |
| (A4) $(\varphi\&(\varphi \to \psi)) \to (\psi\&(\psi \to \varphi))$ |
| (A5a) $((\varphi\&\psi) \to \chi) \to \varphi \to (\psi \to \chi))$ |
| (A5b) $(\varphi \to (\psi \to \chi)) \to ((\varphi\&\psi) \to \chi)$ |
| (A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$ |
| (A7) $\bot \to \varphi$ |
| **Rules** |
| (Mp) $\varphi, (\varphi \to \psi) \vdash \psi$ |

[1]speaker

$$[\varphi] \leq [\psi] \text{ if and only if } \vdash (\varphi \to \psi)\&(\psi \to \varphi)$$

form a lattice. In addition, the following operators on **L**

$$[\varphi] * [\psi] = [\varphi\&\psi] \quad , \quad [\varphi] \dot{\to} [\psi] = [\varphi \to \psi],$$

give an algebra based on the underlying lattice **L**, called BL-algebra. Now, any Bl-algebra could be considered as the set of truth values for continuous t-norm based fuzzy logic.

Propositional Łukasiewicz logic, Gödel logic, and product logic are special cases of continuous t-norm based fuzzy logics. Each of this logics are axiomatized by one or two more axioms added to axioms of BL. The effect of adding this axiom, gives particular classes of BL-algebras, namely MV-algebras, Gödel algebras and product algebras, respectively. MV-algebras and product algebras are closely related to linearly ordered Abelian groups. Indeed, each linearly ordered MV-algebra is a bounded positive cone of some totally ordered Abelian group, and each linearly ordered product logic minus its least element, is the negative cone of some totally ordered Abelian group.

Here, we introduce a logic whose corresponding algebra minus its least and greatest elements is an ordered Abelian group and named it "AG". Since, each such group represented as a subdirect product of totally ordered Abelian groups, we obtain a completeness for this logic. Then, we study the first-order case of this logic.

## 2 Main results

The propositional calculus of AG has propositional variables $\{p1, p2, ...\}$, connectives $\{\wedge, \to, \oplus, \ominus\}$ and truth constants $\perp$ and $\mathfrak{e}$. Formulas are defined in the obvious way. Further connectives are defined as follows:

$$\begin{array}{rcl} 0\varphi & := & \mathfrak{e} \\ n\varphi & := & (n-1)\varphi \oplus \varphi \\ \varphi \vee \psi & := & ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi) \\ \neg\varphi & := & \varphi \to \perp \\ \varphi \leftrightarrow \psi & := & (\varphi \to \psi) \wedge (\psi \to \varphi) \\ \top & := & \neg\perp \end{array}$$

$[0,1]$ is considered as the standard truth value set. An evaluation of propositional variables is a mapping $v$ assigning to each propositional variable $p$ its truth value $v(p) \in [0,1]$. This extends uniquely to the evaluation of all formulae as follows:

- $v(\perp) = 0$,
- $v(\mathfrak{e}) = \frac{1}{2}$,
- $v(\varphi \wedge \psi) = \min\{v(\varphi), v(\psi)\}$,
- $v(\varphi \to \psi) = \begin{cases} 1 & v(\varphi) \leq v(\psi) \\ v(\psi) & v(\varphi) > v(\psi) \end{cases}$,
- $v(\varphi \oplus \psi) = f^{-1}\Big(f\big(v(\varphi)\big) + f\big(v(\psi)\big)\Big)$ where $f(x) = tan(\pi x - \frac{\pi}{2})$,
- $v(\ominus\varphi) = 1 - v(\varphi)$.

Note that if we set $x \star y = f^{-1}\big(f(x) + f(y)\big)$, then one could easily verify that $(0,1)_{AG} = \big((0,1), \star, \frac{1}{2}, \leq\big)$ is a totally ordered Abelian group.

The following formulae are axioms of AG:

Table 2: Axioms of AG

| Axioms of Gödel logic |
|---|
| (G1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$ |
| (G2) $(\varphi \wedge \psi) \to \varphi$ |
| (G3) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$ |
| (G4) $\varphi \to (\varphi \wedge \varphi)$ |
| (G5) $(\varphi \to (\psi \to \chi)) \leftrightarrow ((\varphi \wedge \psi) \to \chi)$ |
| (G6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$ |
| (G7) $\bot \to \varphi$ |
| **Axioms of ordered Abelian group structure** |
| (AG1) $(\varphi \oplus \psi) \leftrightarrow (\psi \oplus \varphi)$ |
| (AG2) $(\neg\neg\varphi \wedge \neg\neg\psi \wedge \neg\neg\theta) \vee \Big(\big(\varphi \oplus (\psi \oplus \theta)\big) \leftrightarrow \big((\varphi \oplus \psi) \oplus \theta\big)\Big)$ |
| (AG3) $(\varphi \oplus \mathfrak{e}) \leftrightarrow \varphi$ |
| (AG4) $\big(\varphi \oplus (\ominus\varphi)\big) \leftrightarrow \mathfrak{e}$ |
| (AG5) $\Big((\varphi \to \psi) \to \big((\varphi \oplus \theta) \to (\psi \oplus \theta)\big) \vee \Big((\psi \oplus \theta) \to \theta\Big)$ |
| (AG6) $\top \to (\ominus\bot)$ |
| (AG7) $(\ominus\top) \to \bot$ |
| (AG8) $\neg\neg\mathfrak{e}$ |
| (AG9) $\neg\neg(\ominus\mathfrak{e})$ |

Note that AG1-AG5 are about the group structure of the truth value set. Specially, observe that AG2 express associative property and AG5 is one of the properties of order in ordered Abelian groups. One could easily verify that all axioms of AG are tautologies.

**Definition 2.1.** Let $T$ be a theory over AG. Two proposition $\varphi$ and $\psi$ are say that to be $T$-equivalent whenever $T \vdash \varphi \leftrightarrow \psi$. For each proposition $\varphi$, let $[\varphi]$ be the equivalence class of $T$-equivalent formulas and assume that $Lind(T)$ be the set of all equivalence classes of $T$-equivalent formulas. Define,

$$[\varphi] + [\psi] = [\varphi \oplus \psi].$$

An easy argument show that $G_{Lind(T)} = \big(Lind(T) - \{[\top], [\bot]\}, +, [\mathfrak{e}]\big)$ is an Abelian group. On the other hand, there as an ordering on $G_{Lind(T)}$ defined by

$$[\varphi] \leq [\psi] \text{ if and only if } T \vdash \varphi \to \psi,$$

which makes $G_{Lind(T)}$ an Abelian lattice ordered group. We call $G_{Lind(T)}$ the Lindenbaum group of $T$-equivalence sentences.

**Definition 2.2.** Let $(G, *, \leq)$ be a lattice ordered Abelian group with an identity element $1_G$ in which for all $a$ and $b$ in $G$ there is a greatest element $x$ of $H$ such that $a \curlywedge x \leq b$ which is denoted by $a \dot{\to} b$. Set $\Gamma_G = \{0\} \cup G \cup \{\infty\}$, and let

$\infty * 0 = 0 * \infty = 1_G$,

for all $a \in G$, $a * \infty = \infty * a = \infty$ and $a * 0 = 0 * a = 0$,

for all $a \in G$, $a \dot{\to} \infty = \infty$, $\infty \dot{\to} a = a$, $a \dot{\to} 0 = 0$, and $0 \dot{\to} a = \infty$,

$0^{-1} = \infty$ and $\infty^{-1} = 0$.

Extend the order $\leq$ on $\Gamma_G$ such that 0 and $\infty$ be the least and largest elements of $\Gamma_G$.

For any lattice ordered Abelian group $G$, $\Gamma_G$ could be considered as the of truth values in AG. precisely, an evaluation is a mapping $v$ from the set of all propositions to $\Gamma_G$ such that:

$$v(\bot) = 0 \quad , \quad v(\mathfrak{e}) = 1_G \quad , \quad v(\varphi \wedge \psi) = v(\varphi) \curlywedge v(\psi) \quad , \quad v(\varphi \oplus \psi) = v(\varphi) * v(\psi),$$
$$v(\ominus\varphi) = \big(v(\varphi)\big)^{-1} \quad , \quad v(\varphi \to \psi) = v(\varphi) \dot{\to} v(\psi).$$

The Lindenbaum group of $T$-equivalence sentences with the following operator

$$[\varphi] \dot{\to} [\psi] = [\varphi \to \psi] \quad , \quad [\varphi] \curlywedge [\psi] = [\varphi \wedge \psi] \quad , \quad [\varphi] + [\psi] = [\varphi \oplus \psi] \quad , \quad -[\varphi] = [\ominus\varphi] \quad , \quad 1 = [\mathfrak{e}],$$

gives a truth value set $\Gamma_{G_{Lind(T)}} = G_{Lind(T)} \cup \{[\top], [\bot]\}$ which is called the Lindenbaum truth-value set of $T$-equivalence sentences.

**Remark 2.3.** By the Lorenzen representation of Abelian lattice ordered groups, each such group is represented as a subdirect product of totally ordered Abelian groups [2].

Now, it is easy to see that the completeness theorem hold in AG.

**Theorem 2.4.** *For any theory $T$ and proposition $\varphi$ over $AG$ the followings are equivalent:*

- $T \vdash \varphi$
- *For each totally ordered Abelian group $G$, $T \models_G \varphi$*
- *For each lattice ordered Abelian group $G$, $T \models_G \varphi$*

In the first-order case, adding the axioms and rules listed in Table 3, we could prove the completeness theorem.

Table 3: Axioms and rules of quantifiers in AG

| Axioms of Gödel logic |
|---|
| $(\forall 1)$ $(\forall x\, \varphi(x)) \to \varphi(t)$ ($t$ substitutable for $x$ in $\varphi(x)$) |
| $(\forall 2)$ $\big(\forall x\, (\psi \to \varphi(x))\big) \to \big(\psi \to (\forall x\, \varphi(x))\big)$ ($x$ not free in $\psi$) |
| $(\forall 3)$ $\big(\forall x\, (\psi \vee \varphi(x))\big) \to \big(\psi \vee (\forall x\, \varphi(x))\big)$ ($x$ not free in $\psi$) |
| $(\exists 1)$ $\varphi(t) \to \big(\exists x\, \varphi(x)\big)$ ($t$ substitutable for $x$ in $\varphi(x)$) |
| $(\exists 2)$ $\big(\forall x\, (\psi \to \varphi(x))\big) \to \big((\exists x\, \varphi(x)) \to \psi\big)$ ($x$ not free in $\psi$) |
| **Rules** |
| (Gen) $\varphi \vdash (\forall x\, \varphi(x))$ |

**Theorem 2.5.** *For any first-order theory $T$ and first-order formula $\varphi$ over $AG$ the followings are equivalent:*

- $T \vdash \varphi$
- *For each lattice ordered Abelian group $G$ and truth value set $\Gamma_G$ and each safe $G$-model [2] $\mathcal{M}$ of $T$, $\varphi^{\mathcal{M}} = \infty$.*

**Corollary 2.6.** *(Compactness Theorem) A theory $T$ is satisfiable if and only if it is finitely satisfiable.*

Note that if $T$ is finitely satisfiable by $G$-models, then it is not necessarily satisfiable by a $G$-model, while by the above corollary $T$ is satisfiable by a $G'$-model for some totally ordered Abelian group $G'$.

**Example 2.7.** Define two new connectives $\prec$ and $\ll$ as follows:

$$\varphi \prec \psi := (\psi \to \varphi) \to \psi \quad , \quad \varphi \ll \psi := ((\varphi \prec \psi) \wedge \neg\neg\psi^{-1}) \vee (\psi \wedge \neg\neg\varphi^{-1}).$$

If $\epsilon$ and $\rho$ be two nullary predicate symbols, then $T = \{\bar{1} \ll \rho, \epsilon \ll \top\} \cup \{\rho^n \ll \epsilon\}_{n\in\mathbb{N}}$ is finitely satisfiable by the standard model, but it has no standard model. Note that an easy calculation show that for any two sentences $\varphi, \psi$ and any structure $\mathcal{M}$, $\mathcal{M} \models \varphi \ll \psi$ if and only if $\varphi^{\mathcal{M}} < \psi^{\mathcal{M}}$.

[2] A safe $G$-model $\mathcal{M}$ is a set $M$ together with interpretations for function and predicate symbols of the underlying language, in which the interpretation of a formula $\varphi(x_1, ..., x_n)$ is a total function $\varphi^{\mathcal{M}} : M^n \to \Gamma_G$.

To develop the model theory assume that $e$ is a binary predicate symbol, whose rule is as the equality relation in classical first-order logic. So, it would satisfy the essential properties of the equality relation, named similarity axioms:

$$\{\forall x\, e(x,x), \forall x \forall y\ (e(x,y) \to e(y,x))\,, \forall x \forall y \forall z\ ((e(x,y) \wedge e(y,z)) \to e(x,z))\}.$$

If a standard model $\mathcal{M}$ satisfy the similarity axioms, then

$$\begin{gathered} e^{\mathcal{M}}(a,a) = 1, \\ e^{\mathcal{M}}(a,b) = e^{\mathcal{M}}(b,a), \\ e^{\mathcal{M}}(a,b) \geq \min\{e^{\mathcal{M}}(a,c), e^{\mathcal{M}}(b,c)\}. \end{gathered}$$

So, the interpretation of $e^{-1}$ in $\mathcal{M}$ is a pseudo-ultrametric on the universe of $\mathcal{M}$, i.e. a pseudo-metric in which for all $a,b,c \in M$, $(e^{-1})^{\mathcal{M}}(a,b) \leq \max\{(e^{-1})^{\mathcal{M}}(a,c), (e^{-1})^{\mathcal{M}}(b,c)\}$. This is the first step in the development of the model theory for AG, which could be done using the ideas in [3, 4].

## References

[1] P. Hájek, *Metamathematics of Fuzzy Logic*, Trends in Logic, Kluwer Academic Publication, 1998.

[2] M. Anderson, T. Feil, *Lattice-Ordered Groups: an introduction*, D. Reidel Publishing Company, 1988.

[3] I. Ben Yaacov, A. Berenstein, C.W. Henson, A. Usvyatsov,*Model theory for metric structures*, Model theory with applications to algebra and analysis, Volume 2, Cambridge Univ Press, 2008.

[4] S. M. A. Khatami, M. Pourmahdian, N. Tavana, *From rational Gödel logic to ultrametric logic*, Journal of Logic and Computation, 26 (2016), pp. 1743–1767.

Email: `khatami@birjandut.ac.ir`