COMPACTNESS OF FIRST-ORDER FUZZY LOGICS

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Abstract. One of the nice properties of the first-order logic is the compactness of satisfiability. It states that a finitely satisfiable theory is satisfiable. Here, some new results are given around the compactness of satisfiability in Hájek Basic logic. It is shown that there are topologies on $[0,1]$ and $[0,1]^2$ for which the interpretation of all logical connectives of the Basic logic are continuous. Furthermore, a topology on first-order structures is introduced for any similarity relation, and then by the same ideas as in the continuous logic, the results around the compactness of satisfiability for Basic logic are extended.

1. Introduction

In this paper, we shall study the compactness of satisfiability for the Hájek Basic logic, **BL**. In many cases, in fact, the usual compactness fails in special case of **BL** such as in the Gödel and Product logic whose set of truth values is the continuous scale $[0,1]$. In spite of these special cases, however, changing the truth value set or generalizing the concept of satisfiability to K-satisfiability leads to some version of the compactness.

Here, the ideas in [CN04], [CK66], [TPD12], and [KP15] are extended to derive new results around the usual compactness and K-compactness of **BL** and **BL**$\forall$. Besides improving the results around compactness, some topologies on both of truth value sets and structures are introduced which may be interesting for study.

It seem's that the only fuzzy logic that satisfies the usual compactness as well as the K-compactness for any compact subset K of the unit interval $[0,1]$, in both propositional and first-order cases, is the Łukasiewicz logic [CN04, BKZ95, TPD12]. In the case of propositional Łukasiewicz logic an easy application of the Tychonoff theorem leads to the result [CN04, BKZ95]. In the first-order case, there are several methods, of which the most significant one is the "Ultraproduct method" [TPD12, Del14]. In fact, the main reason behind this is the continuity of the truth function of logical connectives of the Łukasiewicz logic with respect to the Euclidean topology on $[0,1]$. However, we show that for any continuous t-norm $*$, if one consider a topology T_* on $[0,1]$ whose base is the collection of balls of the form $B_r(a) = \{b : e(a,b) > r\}$ in which e is the interpretation of $\leftrightarrow$, then T_* induced a topology $\mathbf{T}_*$ on $[0,1]^2$ such that the interpretation of all of logical connectives becomes continuous functions. Using this fact, by the same ways as [CN04] a version of K-compactness derived for **BL**. In addition, for any similarity relation ρ, a topology on ant first-order structure is introduced which is called the similarity topology and then using the introduced methods of Chang and Keisler [CK66],the K-compactness for **BL**$\forall$ is obtained.

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2. Preliminaries

2.1. **Syntax.** Basic logical connectives of Basic logic are $\{\&, \to, \bot\}$. Basic logic can be presented by a semantics on BL-algebras.

Recall that a BL-algebra is an algebra $\mathbf{L} = (L, \cap, \cup, \star, \rightarrowtail, 0_{\mathbf{L}}, 1_{\mathbf{L}})$ of type $(2, 2, 2, 2, 0, 0)$ such that $(L, \cap, \cup, 0_{\mathbf{L}}, 1_{\mathbf{L}})$ is a bounded lattice with greatest element $1_{\mathbf{L}}$ and smallest element $0_{\mathbf{L}}$, $(L, \star, 1_{\mathbf{L}})$ is an Abelian monoid, $\rightarrowtail$ is the residua of $\star$ that is "$c \leq a \rightarrowtail b$ iff $c \star a \leq b$ for all $a, b, c \in L$", L is pre-linear that is "$(a \rightarrowtail b) \cup (b \rightarrowtail a) = 1$ for all $a, b \in L$", and finally $a \cap b = a \star (a \rightarrowtail b)$ for all $a, b \in L$ [Háj98, Chapter 2]. When $[0, 1]_\star = ([0, 1], \cap, \cup, \star, \rightarrowtail, 0, 1)$ endowed to be a BL-algebra, then the binary operator $\star$ becomes a continuous t-norm on $[0, 1]$ and $\rightarrowtail$ becomes the residua of $\star$ that is $x \rightarrowtail y = \max\{z : z \star x \leq y\}$ [BEG99]. For a continuous t-norm $*$ the residua of $*$ is denoted by $\Rightarrow_*$.

Given a first-order language $\mathcal{L}$ consisting of function and predicate symbols and a BL-algebra $\mathbf{L}$, an $\mathbf{L}$-structure $\mathcal{M}$ for $\mathcal{L}$ is a nonempty set M together with interpretations of functions by $f_{\mathcal{M}} : M^n \to M$ and interpretations of predicates as $P_{\mathcal{M}} : M^n \to L$.

2.2. **Semantics.** For a BL-algebra $\mathbf{L}$, $\bot$ is interpreted by $0_{\mathbf{L}}$, & is interpreted by $\star$, and $\to$ is interpreted by $\rightarrowtail$. So, in the standard semantics of **BL** on $[0, 1]_\star$, the strong conjunction & is interpreted by a continuous t-norm $*$, the implication is interpreted by $\Rightarrow_*$ and the zero function plays the role of $\bot$.

any mapping v from the set of variables into the underlying universe of $\mathcal{M}$ is called an $\mathcal{M}$-evaluation. Interpretation of terms, formulas and sentences are defined inductively as usual [Háj98, Chapter 2], and denoted by $\|t(\bar{x})\|_{\mathcal{M},v}$, $\|\varphi(\bar{x})\|^{\mathbf{L}}_{\mathcal{M},v}$, and $\|\varphi\|^{\mathbf{L}}_{\mathcal{M}}$. For $[0, 1]_*$-structure $\mathcal{M}$ and $\mathcal{M}$-evaluation v, we use $\|\varphi\|_{\mathcal{M}}$ and $\|\varphi\|_{\mathcal{M},v}$ instead of $\|\varphi\|^{[0,1]_*}_{\mathcal{M}}$ and $\|\varphi\|^{[0,1]_*}_{\mathcal{M},v}$.

Let $*$ be a continuous t-norm, $\mathbf{L}$ be a subalgebra of $[0, 1]_*$, and $K \subseteq L \subseteq [0, 1]$. A proposition φ is said to be $K^{\mathbf{L}}$-satisfiable if there exists an $\mathbf{L}$-evaluation v ($\mathbf{L}$-model $\mathcal{M}$) such that $v(\varphi) \in K$ ($\|\varphi\|^{\mathbf{L}}_{\mathcal{M}} \in K$). In this way, v ($\mathcal{M}$) is called a $K^{\mathbf{L}}$-model of φ. In the case that $\mathbf{L} = [0, 1]_*$, φ is called a K-satisfiable theory. A theory whose propositions are satisfied by a $K^{\mathbf{L}}$-model (K-model) v, is called a $K^{\mathbf{L}}$-satisfiable (K-satisfiable) theory.

2.3. **Known results.**

- Let K be a compact subset of $[0, 1]$ with the Euclidean topology. Every finitely K-satisfiable theory over **Ł** is K-satisfiable.
- Let K be a noncompact subset of $[0, 1]$ with the Euclidean topology. There is a finitely K-satisfiable theory over **Ł** such that it is not K-satisfiable.
- Let K be a compact subset of $[0, 1]$ with the Euclidean topology. Every finitely K-satisfiable theory over **Ł**$\forall$ is K-satisfiable.

The main reason behind this facts is the continuity of the interpretation of logical connectives in **Ł** and **Ł**$\forall$. The non-continuity of the interpretation of the implication connective in Gödel logic as well as Product logic, break down getting a general result about the compactness in these logics. However, some partial results are obtained in the literature.

- Assume that the set of atomic propositions is at most countable. Then every finitely 1-satisfiable theory over **G** is 1-satisfiable.
- Assume that $\mathcal{L}$ is an at most countable first-order language. Every finitely 1-satisfiable $\mathcal{L}$-theory over **G**$\forall$ is 1-satisfiable.
- Let K be a finite subset of $[0, 1]$. **G** with at most countable set of atomic propositions and **G**$\forall$ with at most countable underlying language are K-compact.

- Let $\mathcal{L}$ be an at most countable first-order language and K be a closed subset of $[0,1]$. $\mathbf{G}\forall$ is not K-compact if and only if K is infinite and $1 \notin K$.
- Assume that $K \subseteq (0,1]$ contains 1. Then $\mathbf{G}$ and $\mathbf{\Pi}$ are K-compact.

3. Main results

3.1. **$*$-Open Ball Topology.** In this subsection, for any subalgebra $\mathbf{L}$ of the standard BL-algebra $[0,1]_*$, two topologies on L and L^2 are introduced for which all of the operators of $\mathbf{L}$ becomes continuous functions. From now on assume that e_* be the interpretation of the equivalence relation, i.e., $e_* : L^2 \to L$ is defined by $e_*(a,b) = (a \Rightarrow_* b) * (b \Rightarrow_* a)$. Furthermore, let the binary operator $\mathbf{e}_*$ on L^2 is defined by $\mathbf{e}_*(\bar{a},\bar{b}) = e_*(a_1,b_1) * e_*(a_2,b_2)$ in which $\bar{a} = (a_1,a_2)$ and $\bar{b} = (b_1,b_2)$.

Definition 3.1. Let $*$ be a continuous t-norm and $\mathbf{L} = (L,\cap,\cup,*,\Rightarrow_*,0_{\mathbf{L}},1_{\mathbf{L}})$ be a subalgebra of $[0,1]_*$. For any $a \in L$ ($\bar{a} \in L^2$) and $r \in L \setminus \{1_{\mathbf{L}}\}$ the set $B_r(a) = \{b \in L : e_*(a,b) > r\}$ ($\mathbf{B}_r(\bar{a}) = \{\bar{b} \in L^2 : \mathbf{e}_*(\bar{a},\bar{b}) > r\}$) is called the $*$-ball around a ($*$-ball around $\bar{a}$) of radius r. A subset G of L (L^2) is called a $*$-open set if for every $a \in G$ ($\bar{a} \in G$) there exists $r \in L \setminus \{1_{\mathbf{L}}\}$ such that $B_r(a) \subseteq G$ ($\mathbf{B}_r(\bar{a}) \subseteq G$). The $*$-open ball topology on L (L^2) is $T_* = \{G : G \text{ is a } *\text{-open subset of } L\}$ ($\mathbf{T}_* = \{G : G \text{ is a } *\text{-open subset of } L^2\}$).

Theorem 3.2. *Let $*$ be a continuous t-norm and $\mathbf{L}$ be a subalgebra of $[0,1]_*$. The sets T_* and $\mathbf{T}_*$ form topologies on L and L^2. Furthermore, (L,T_*) is a Hausdorff space.*

Theorem 3.3. *Let $*$ be a continuous t-norm and $\mathbf{L} = (L,\cap,\cup,*,\Rightarrow_*,0_{\mathbf{L}},1_{\mathbf{L}})$ be a subalgebra of $[0,1]_*$. Then the functions $* : (L^2,\mathbf{T}_*) \to (L,T_*)$ and $\Rightarrow_* : (L^2,\mathbf{T}_*) \to (L,T_*)$ are continuous functions.*

3.2. **Similarity topology.** Assume that the underlying language of any theory contains a binary predicate symbol ρ as the similarity relation. The axioms of similarity are as follows [Háj98, Chapter 5].

(S1) $\forall x \rho(x,x)$.
(S2) $\forall x \forall y \big(\rho(x,y) \to \rho(y,x)\big)$.
(S3) $\forall x \forall y \forall z \big[\big(\rho(x,y) \& \rho(y,z)\big) \to \rho(x,z)\big]$.

For a similarity relation ρ, the binary n-tuple similarity relation is defined by

$$\rho_n(\bar{x},\bar{y}) = \rho(x_1,y_1) \& \rho(x_2,y_2) \& \ldots \& \rho(x_n,y_n).$$

Exitance of a similarity relation on an structure, induced a topology on it.

Definition 3.4. Let $*$ be a continuous t-norm, $\mathbf{L}$ be a subalgebra of $[0,1]_*$, and $\mathcal{M}$ be an $\mathbf{L}$-structure satisfying the axioms of similarity. For $r \in [0_{\mathbf{L}},1_{\mathbf{L}})$ and $a \in M$ the ρ-open ball around a is $B^{\rho}_r(a) = \{m \in M : \|\rho(a,m)\|^{\mathbf{L}}_{\mathcal{M}} > r\}$. For $r \in [0_{\mathbf{L}},1_{\mathbf{L}})$ and $\bar{a} \in M^n$ the ρ-open ball around $\bar{a}$ is $B^{\rho}_r(\bar{a}) = \{\bar{m} \in M^n : \|\rho_n(\bar{a},\bar{m})\|^{\mathbf{L}}_{\mathcal{M}} > r\}$. A subset $G \subseteq M$ ($G \subseteq M^n$) is called ρ-open if for each $a \in G$ ($\bar{a} \in G$) there exists $r \in [0_{\mathbf{L}},1_{\mathbf{L}})$ such that $B^{\rho}_r(a) \subseteq G$ ($B^{\rho}_r(\bar{a}) \subseteq G$). The similarity topologies on M and M^n are the set of all ρ-open subsets of M and M^n which are denoted by T_ρ and T^n_ρ, respectively.

Theorem 3.5. *Let $*$ be a continuous t-norm, $\mathbf{L}$ be a subalgebra of $[0,1]_*$, and $\mathcal{M}$ be an $\mathbf{L}$-structure satisfying the axioms of similarity. T_ρ and T^n_ρ form topologies on M and M^n.*

For any n-ary predicate symbol P and n-ary function symbol f, the extensionality axioms with respect to the similarity relation ρ are as follows [Háj98, Chapter 5].

(E1) $\forall \bar{x}, \forall \bar{y} \big[\rho_n(\bar{x},\bar{y}) \to \big(P(\bar{x}) \leftrightarrow P(\bar{y})\big)\big]$.

(E2) $\forall\bar{x}, \forall\bar{y}\big[\rho_n(\bar{x},\bar{y}) \to \rho\big(f(\bar{x}), f(\bar{y})\big)\big]$.

Theorem 3.6. *Let $*$ be a continuous t-norm, $\mathbf{L}$ be a subalgebra of $[0,1]_*$, $\mathcal{L}$ be a first-order language, and $\mathcal{M}$ be an $\mathbf{L}$-structure for $\mathcal{L}$ satisfying the axioms of similarity. If for some n-ary predicate symbol $P \in \mathcal{L}$ and some n-ary function symbol $f \in \mathcal{L}$, $\mathcal{M}$ satisfies (E1) and (E2), then $P_{\mathcal{M}} : (M^n, T^n_\rho) \to (\mathbf{L}, T_*)$ and $f_{\mathcal{M}} : (M^n, T^n_\rho) \to (M, T_\rho)$ are continuous functions.*

Definition 3.7. Let $*$ be a continuous t-norm, $\mathbf{L}$ be a subalgebra of $[0,1]_*$, and $\mathcal{L}$ be a first-order language containing a similarity relation ρ. An $\mathbf{L}$-structure $\mathcal{M}$ for $\mathcal{L}$ which satisfies the similarity axioms for ρ, and furthermore, satisfies the extensionality axioms with respect to the similarity relation ρ for every function symbol and predicate symbol, is called a continuous $\mathbf{L}$-structure for $\mathcal{L}$.

3.3. **Compactness results.** Now, in the propositional Basic logic, K-compactness could be proved as in the propositional Łukasiewicz logic [BKZ95, CN04].

Theorem 3.8. *Let $*$ be a continuous t-norm, $\mathbf{L}$ be a subalgebra of $[0,1]_*$, (L, T_*) be a compact topological space, and K be compact subset of (L, T_*). Then in the Basic logic every finitely $K^{\mathbf{L}}$-satisfiable theory over $\mathbf{BL}$ is $K^{\mathbf{L}}$-satisfiable.*

Furthermore, using an ultraproduct construction such as introduced in [CK66] or [TPD12] we derive the K-compactness for $\mathbf{BL}\forall$.

Theorem 3.9. *(Compactness theorem) Let $*$ be a continuous t-norm, $\mathbf{L}$ be a subalgebra of $[0,1]_*$, (L, T_*) be compact space, and K be a compact subset of (L, T_*). Then, every finitely $K^{\mathbf{L}}$-satisfiable theory over $\mathbf{BL}\forall$ by continuous structures is $K^{\mathbf{L}}$-satisfiable.*

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