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Compactness property of predicate fuzzy logics

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Abstract

The compactness property in classical logic state that a finitely satisfiable theory is satisfiable. In many-valued logics, this property does not hold in general cases. However, the compactness theorem is proved in some cases such as the Łukasiewicz logic. Here we have an overview on some results about the usual compactness theorem and then we extend the results to predicate Gödel logic and product logic.

Keywords: compactness theorem, many-valued logics, fuzzy logics, continuous t-norm
Mathematics Subject Classification [2010]: 03B50, 03C20

1 Introduction

While dealing with an uncertain information, one often switch from bivalent of the truth value set to many-valent. This poses different kinds of many-valued logics as well as various kinds of the compactness property of these logics. The class of all many-valued logics is very large to study. However, as the metamathematics of continuous t-norm based fuzzy logics have been studied in [1], we shall study the compactness property of these logics.

Fuzzy logic arises by assigning degrees of truth to propositions. The standard set of truth values is the basic unit interval of real numbers, $[0, 1]$, where 0 represents totally false, 1 represents totally true, and the other numbers refer to partial truth, i.e., intermediate degrees of truth. A continuous t-norm based fuzzy logics is a logic with the primitive connectives $\{\&, \to, \bot\}$ and derivable connectives defined as:

$$\neg\varphi := \varphi \to \bar{1} \quad , \quad \top := \neg\bot \quad , \quad \varphi \wedge \psi := \varphi\&(\varphi \to \psi).$$
$$\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi) \quad , \quad \varphi \leftrightarrow \psi := (\varphi \to \psi)\&(\psi \to \varphi).$$

The semantics of propositional fuzzy logics based on continuous t-norm given by a continuous t-norms. Namely, given a particular continuous t-norm $*$, an evaluation e is a mapping from propositional variables to $[0, 1]$, extended to all formulas by interpreting & as $*$, the implication $\to$ as the residuum of $*$, and $\bot$ as 0, respectively. For propositional fuzzy logics based on continuous t-norms, a systematic study have been done for the usual compactness as well as the K-compactness in [2].

In the case of predicate fuzzy logics, for a given first-order language $\mathcal{L}$, the concept of $\mathcal{L}$-structure is defined as in classical logic. The interpretation of $\mathcal{L}$-formula $\varphi(\bar{x})$ is a function $\varphi^{\mathcal{M}} : M^n \to [0, 1]$ which is inductively defined as follows:

$\bot^{\mathcal{M}} = 0$.
for every n-ary predicate symbol P, $P(t_1, ..., t_n)^{\mathcal{M}}(\bar{a}) = P^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), ..., t_n^{\mathcal{M}}(\bar{a}))$,
$(\varphi\&\psi)^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{M}}(\bar{a}) * \psi^{\mathcal{M}}(\bar{a})$,
$(\varphi \to \psi)^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{M}}(\bar{a}) \Rightarrow \psi^{\mathcal{M}}(\bar{a})$, where $\Rightarrow$ is the residuum of t-norm $*$,
if $\varphi(\bar{x}) = \forall y\ \psi(y, \bar{x})$ then $\varphi^{\mathcal{M}}(\bar{a}) = \inf_{b\in M}\{\psi^{\mathcal{M}}(b, \bar{a})\}$,
if $\varphi(\bar{x}) = \exists y\ \psi(y, \bar{x})$ then $\varphi^{\mathcal{M}}(\bar{a}) = \sup_{b\in M}\{\psi^{\mathcal{M}}(b, \bar{a})\}$.

[1] speaker

There is no such a systematic study for the compactness property of predicate fuzzy logics, as done in [2] for propositional fuzzy logics. A recent result by Tavana et al. in [3] show that the first-order Łukasiewicz logic admit the compactness property as well as the K-compactness for any closed subset K of the unite interval $[0,1]$. Another study for the Gödel logics by Pourmahdian and Tavana [4] show that in some cases Gödel logics admit the compactness property. But, in many cases, in fact,even the usual compactness fail in predicate fuzzy logics. The following two examples show that the usual compactness theorem fail in first-order product logics and Gödel logics whose set of truth values is the continuous scale $[0,1]$.

Example 1.1. Let $\mathcal{L}$ be a relational language containing two unary predicate symbols $\{R(x),\rho(x)\}$. Let

$$T=\Big\{\neg\forall x\,\big(R(x)\vee\rho(x)\big),\neg\neg\forall x\,R(x),\forall x\,\big(R(x)\to\rho^n(x)\big)\Big\}.$$

If (in product logic) $\mathcal{M}\models T$, then $\mathcal{M}\models\neg\forall x\,\big(R(x)\vee\rho(x)\big)$ and so there is an element $b\in M$ such that $\max\{R^{\mathcal{M}}(b),\rho^{\mathcal{M}}(b)\}<1$. Furthermore, since $\mathcal{M}\models\neg\neg\forall x\,R(x)$ for all $a\in M$, $R^{\mathcal{M}}(a)>0$ and specially $0<R^{\mathcal{M}}(b)<1$. On the other hand, as for each $n\geq 1$, $\mathcal{M}\models\forall x\,\big(R(x)\to\rho^n(x)\big)$,

$$\inf_{a\in M}\big(R^{\mathcal{M}}(a)\to(\rho^n)^{\mathcal{M}}(a)\big)=1.$$

In particular $\big(R^{\mathcal{M}}(b)\to(\rho^n)^{\mathcal{M}}(b)\big)=1$. Therefore $0<R^{\mathcal{M}}(b)\leq(\rho^n)^{\mathcal{M}}(b)<\rho^{\mathcal{M}}(b)<1$ for all $n\geq 1$, which is impossible. Thus T is not satisfiable. However, obviously T is finitely satisfiable.

Example 1.2. Consider the standard Gödel logic with truth value set $[0,1]$. Let $\mathcal{L}$ be a relational language contains uncountably many unary predicate symbols $\{R(x)\}\cup\{\rho_i(x)\}_{i\in\omega_2}$. Set,

$$T=\Big\{\neg\forall x\,R(x),\forall x\,\Big(\big(R(x)\to\rho_1(x)\big)\to R(x)\Big)\Big\}\cup\Big\{\forall x\,\Big(\big(\rho_j(x)\to\rho_i(x)\big)\to\rho_j(x)\Big):i>j\Big\}_{i,j\in\omega_2}$$

If $\mathcal{M}\models T$, then $\mathcal{M}\models\neg\forall x\,R(x)$ and so there is an element $a\in M$ such that $R^{\mathcal{M}}(a)<1$. On the other hand, as $\mathcal{M}\models\forall x\,\Big(\big(R(x)\to\rho_1(x)\big)\to R(x)\Big)$, thus

$$\rho_1^{\mathcal{M}}(a)<R^{\mathcal{M}}(a)<1.$$

Furthermore, for every $i>j\in\omega_2$, $\mathcal{M}\models\forall x\,\Big(\big(\rho_j(x)\to\rho_i(x)\big)\to\rho_j(x)\Big)$. So, we have

$$\rho_{\omega_2}^{\mathcal{M}}(a)<...<\rho_2^{\mathcal{M}}(a)<\rho_1^{\mathcal{M}}(a)<R^{\mathcal{M}}(a)<1,$$

a contradiction with cardinality of $[0,1]$. But, one can easily verify that T is finitely satisfiable.

In spite of these examples, however, changing the truth value set or generalizing the concept of satisfiability to K-satisfiability, leads to some version of compactness in these logics.

Here we extends the ideas in [3] and [5] for a systematic study around the usual compactness theorem in predicate fuzzy logics.

The main reason that ultraproduct method works well for the Łukasiewicz logic is the continuity of the truth function of connectives with respect to the Euclidean topology on the standard truth value set $[0,1]$. On the other hand, one can easily verify that the truth function of $\neg(p\leftrightarrow q)$ in Łukasiewicz logic is the Euclidean metric $d_e(p,q)=|p-q|$, while in Gödel logic or product logic this gives only the discrete metric.

We consider a reverse semantical meaning on the set of truth values which leads to obtain nontrivial metrics in the case of Gödel and product logic. Then using this metric and the ultraproduct method, we prove the usual compactness theorem in these logics.

2 Main results

Let V be a closed subset of $[0,1]$ with the Euclidean topology which is contained both 0 and 1. Consider a reverse semantical meaning on the set of truth values V, i.e., 0 is stands for absolute truth and 1 for absolute falsity. Since 0 is considered for absolute truth, some changes in the interpretation of formulas occur. For example, $\varphi\&\psi$ is completely true whenever both of φ and ψ are completely true, i.e., the maximum of them must be completely true. This is done by considering continuous t-conorms instead of t-norm for the truth-functionality of &. A continuous t-conorm is a binary operation $S: V^2 \to V$ which is continuous with respect to the Euclidean topology, commutative, associative, non-decreasing on both arguments, $S(0,x) = x$ for all $x \in V$, and $S(1,x) = 1$ for all $x \in V$. The residuum of t-conorm S is a binary operation $\dot{\to}: V^2 \to V$ defined by the following property,

$$\text{for all } x, y, z \in \mathbb{D},\ S(z,x) \geq y \text{ iff } z \geq x \dot{\to} y.$$

Continuity of S and closedness of V together imply that $x \dot{\to} y = \min\{z : S(z,x) \geq y\}$. The most important t-conorms and their residuum are listed as follows:

- Łukasiewicz : $S_L(x,y) = \min\{x+y, 1\}$, $x \dot{\to}_L y = \begin{cases} 0 & x \geq y \\ y-x & x < y \end{cases}$,
- Gödel : $S_G(x,y) = \max\{x,y\}$, $x \dot{\to}_G y = \begin{cases} 0 & x \geq y \\ y & x < y \end{cases}$,
- product : $S_\pi(x,y) = x + y - xy$, $x \dot{\to}_\pi y = \begin{cases} 0 & x \geq y \\ \dfrac{y-x}{1-x} & x < y \end{cases}$.

For a given first-order language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{M}$, the interpretation of $\mathcal{L}$-formula $\varphi(\bar{x})$ is a function $\varphi^{\mathcal{M}} : M^n \to V$ which is inductively defined as follows:

$\perp^{\mathcal{M}} = 1$.
for every n-ary predicate symbol P, $P(t_1,...,t_n)^{\mathcal{M}}(\bar{a}) = P^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}),...,t_n^{\mathcal{M}}(\bar{a}))$,
$(\varphi\&\psi)^{\mathcal{M}}(\bar{a}) = S\big(\varphi^{\mathcal{M}}(\bar{a}), \psi^{\mathcal{M}}(\bar{a})\big)$, where S is a continuous t-conorm,
$(\varphi \to \psi)^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{M}}(\bar{a}) \dot{\to} \psi^{\mathcal{M}}(\bar{a})$, where $\dot{\to}$ is the residuum of t-conorm S,
if $\varphi(\bar{x}) = \forall y\ \psi(y,\bar{x})$ then $\varphi^{\mathcal{M}}(\bar{a}) = \sup_{b\in M}\{\psi^{\mathcal{M}}(b,\bar{a})\}$,
if $\varphi(\bar{x}) = \exists y\ \psi(y,\bar{x})$ then $\varphi^{\mathcal{M}}(\bar{a}) = \inf_{b\in M}\{\psi^{\mathcal{M}}(b,\bar{a})\}$.

the truth function of the equivalence connective in Łukasiewicz logic becomes the Euclidean metric $d_e(p,q) = |p-q|$, while in Gödel logic it's truth function is the metric $d_{max} : [0,1]^2 \to [0,1]$ defined by

$$d_{max}(p,q) = \begin{cases} \max\{p,q\} & p \neq q \\ 0 & p = q \end{cases}$$

and in product logic it's truth function is the metric $d_\pi : [0,1]^2 \to [0,1]$ defined by

$$d_\pi(p,q) = \begin{cases} \dfrac{|p-q|}{1-\min\{p,q\}} & p \neq q \\ 0 & p = q \end{cases}.$$

Thus we call this semantic "metric semantic" of fuzzy logics. Note that the function $U : V \to V$ defined by $u(x) = 1 - x$ translate all semantical issues of metric semantic to the usual semantic of fuzzy logics.

Lemma 2.1. *Truth functions of all logical connectives in Gödel logic and product logic as well as the Lukasiewicz logic, are continuous functions from $(V^2, \mathbf{d})$ into (V, d) where d is the metric [2] induced from the truth function of the equivalence connective of the corresponding logic in metric semantic.*

[2] if $d : V^2 \to V$ is a metric, then it is induces a metric $\mathbf{d} : (V^n)^2 \to V^n$ which is defined by $\mathbf{d}\big((a_1,...,a_n),(b_1,...,b_n)\big) = \max\big\{d(a_1,b_1),...,d(a_n,b_n)\big\}$.

We review some facts about filters to prove the compactness theorem using the ultraproduct method.

(Fact1) A filter $\mathfrak{D}$ on a topological space X, convergent to an element $x \in X$, whenever for each open set U containing x, U is an element of $\mathfrak{D}$. This is denoted by $\mathfrak{D} \to x$ and x is called a limit point of $\mathfrak{D}$.

(Fact2) X is a compact Hausdorff space if and only if every filter $\mathcal{D}$ on X has a unique limit point.

(Fact3) Let $f : X \to Y$ be a continuous function at $x_0 \in X$ and $\mathfrak{D}$ be a filter on X. If $f(\mathfrak{D})$ be the filter on Y generated by the set $\{f(A) : A \in \mathfrak{D}\}$, then $f(\mathfrak{D}) \to f(x_0)$.

Definition 2.2. Let X be a topological space, I be a nonempty set, and $\mathfrak{D}$ be a filter on I. Furthermore, let $f \in I^X$, $\{x_i\}_{i\in I}$ be the range of f, and $f^*(\mathfrak{D}) = \{A \subseteq X : f^{-1}(A) \in \mathfrak{D}\}$. If $f^*(\mathfrak{D})$ is convergent to $x \in X$, then we call x the $\mathfrak{D}$-limit of the family $\{x_i\}_{i\in I}$ and write $\lim_{\mathfrak{D}} x_i = x$.

Now, the following corollary is an easy consequence of (Fact3).

Corollary 2.3. *Let $f : X \to Y$ be a continuous function at $x_0 \in X$, I be a nonempty set, and $\mathfrak{D}$ be a filter on I. If $\lim_{\mathfrak{D}} x_i = x_0$ then $\lim_{\mathfrak{D}} f(x_i) = f(x_0)$.*

Lemma 2.4. *Let V be a truth value set, I be a nonempty set, and $\mathfrak{D}$ be a filter on I. For $d \in \{d_e, d_{max}, d_\pi\}$ consider (V, d) where as a topological space. If $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ are two family of elements of V, then*

$$\lim_{\mathfrak{D}} x_i \leq \lim_{\mathfrak{D}} y_i \text{ if and only if } \{i : x_i \leq y_i\} \in \mathfrak{D}.$$

Now, by (Fact2), we could construct the ultraproduct of a family of structures in the first-order Gödel logic and product logic as well as the Łukasiewicz logic.

Definition 2.5. Let V be a truth value set and $d \in \{d_e, d_{max}, d_\pi\}$ and (V, d) is a compact Hausdorff subspace of $([0, 1], d)$. Let $\{\mathcal{M}_i\}_{i\in I}$ be a family of $\mathcal{L}$-structures and $\mathfrak{D}$ be a filter on I. The $\mathfrak{D}$-ultraproduct of family $\{\mathcal{M}_i\}_{i\in I}$ is an $\mathcal{L}$-structure $\mathcal{M}$ with universe $M = \prod_{i\in I} M_i$ whose interpretation of elements of $\mathcal{L}$ is defined as follows.

- For n-ary predicate symbol $R \in \mathcal{L}$, $R^{\mathcal{M}} : M^n \to V$ is defined by

$$R^{\mathcal{M}}\big(\{x_i^1\}_{i\in I}, ..., \{x_i^n\}_{i\in I}\big) = \lim_{\mathfrak{D}} R^{\mathcal{M}_i}(x_i^1, ..., x_i^n).$$

- For n-ary function symbol $f \in \mathcal{L}$, $f^{\mathcal{M}} : M^n \to M$ is defined by

$$f^{\mathcal{M}}\big(\{x_i^1\}_{i\in I}, ..., \{x_i^n\}_{i\in I}\big) = \{f^{\mathcal{M}_i}(x_i^1, ..., x_i^n)\}_{i\in I}.$$

Theorem 2.6. *(Loś theorem) Let V be a truth value set and $d \in \{d_e, d_{max}, d_\pi\}$ and (V, d) is a compact Hausdorff subspace of $([0, 1], d)$. Furthermore, assume that $\{\mathcal{M}_i\}_{i\in I}$ be a family of $\mathcal{L}$-structures. If $\mathfrak{D}$ is an ultrafilter on I and $\mathcal{M}$ is the $\mathfrak{D}$-ultraproduct of family $\{\mathcal{M}_i\}_{i\in I}$, then in the corresponding predicate logic of the ,metric d, for each $\mathcal{L}$-formula $\varphi(x_1, ..., x_n)$ and each $\mathbf{a}_k = \{a_i^k\}_{i\in I} \in M$ $(1 \leq k \leq n)$,*

$$\varphi^{\mathcal{M}}(\mathbf{a}_1, ..., \mathbf{a}_n) = \lim_{\mathfrak{D}} \varphi^{\mathcal{M}_i}(a_i^1, ..., a_i^n).$$

Theorem 2.7. *(Compactness theorem) Let V be a truth value set and $d \in \{d_e, d_{max}, d_\pi\}$ and (V, d) is a compact Hausdorff subspace of $([0, 1], d)$. In the corresponding logic of metric d, every finitely satisfiable theory is satisfiable.*

Proof. Assume that T is a finitely satisfiable theory. Let I be the set of all finite subsets of T. For each $\varphi \in T$, let $\overline{\varphi} = \{\Sigma : \varphi \in \Sigma \text{ and } \Sigma \in I\}$. Obviously $\mathfrak{T} = \{\overline{\varphi} : \varphi \in T\}$ has the finite intersection property. So, there exists an ultrafilter $\mathfrak{D}$ on I containing $\mathfrak{T}$.

Let $T_i \in I$. As T is finitely satisfiable, there exists a structure $\mathcal{M}_i \models T_i$. Suppose that $\mathcal{M}$ be the $\mathfrak{D}$-ultraproduct of $\{\mathcal{M}_i\}_{i\in I}$. By Loś theorem, $\mathcal{M} \models T$. □

In the case of Łukasiewicz logic the standard truth value set $[0,1]$ is a compact Hausdorff topological space with respect to the Euclidean metric d_e. So, the compactness theorem for this logic is hold by considering the standard truth value set $[0,1]$. However, this is not true for Gödel logic or product logic. In the case of Gödel logic for example $V_{\downarrow} = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is a compact Hausdorff subspace of $([0,1], d_{max})$ and so when $V_{\downarrow}$ is considered as the truth value set, Gödel logic admit the compactness theorem. Observe that, if $V_{\downarrow}$ is considered as the truth value set, then theory T in example 1.2 is not finitely satisfiable in metric semantic. In product logic the set $V_p = \{1\} \cup [0, 0.9]$ is the case and the compactness theorem hold in this logic with V_p as the truth value set. In example 1.1 if V_p is considered as truth value set, theory T is not finitely satisfiable in metric semantic.

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