International Conference on Mathematics of Fuzziness
27-29 April 2016, IASBS, Zanjan, Iran.

A NOTE ON THE COMPACTNESS THEOREM FOR FIRST-ORDER GÖDEL LOGICS

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Abstract. By considering a reverse semantical meaning on the truth value set $V_{\downarrow} = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, we show that for a given first-order language $\mathcal{L}$ of any cardinality, the compactness theorem holds in the corresponding Gödel logic.

1. Introduction

The known proof of the compactness theorem in Gödel logic is based on constructing a Gödel algebra by Henkin construction and embedding the corresponding Gödel algebra into the standard truth value set $[0,1]$ [3]. However, this method only works for theories with countable underlying first-order language. The following example show that the compactness theorem in first-order Gödel logic may fail for uncountable languages.

Example 1.1. [1] *Consider the standard Gödel logic with truth value set* $[0,1]$. *Also assume the every-day semantic in which* 1 *is the absolute truth and* 0 *is the absolute falsity. Let* $\mathcal{L}$ *be a relational language contains uncountably many unary predicate symbols* $\{R(x)\} \cup \{\rho_i(x)\}_{i\in\omega_2}$. *Set,*

$$\begin{aligned} T &= \Big\{\neg\forall x\, R(x), \forall x\Big(\big(R(x) \to \rho_1(x)\big) \to R(x)\Big)\Big\} \\ &\cup \Big\{\forall x\Big(\big(\rho_j(x) \to \rho_i(x)\big) \to \rho_j(x)\Big) : i > j\Big\}_{i,j\in\omega_2} \end{aligned}$$

If $\mathcal{M} \models T$, *then* $\mathcal{M} \models \neg\forall x\, R(x)$ *and so there is an element* $a \in M$ *such that* $R^{\mathcal{M}}(a) < 1$. *On the other hand, as*

$$\mathcal{M} \models \forall x\Big(\big(R(x) \to \rho_1(x)\big) \to R(x)\Big),$$

thus

2010 *Mathematics Subject Classification.* Primary 03B50; Secondary 03C20.
Key words and phrases. Gödel logic, Compactness theorem.

$$\rho_1^{\mathcal{M}}(a) < R^{\mathcal{M}}(a) < 1.$$

Furthermore, for every $i > j \in \omega_2$,

$$\mathcal{M} \models \forall x \Big(\big(\rho_j(x) \to \rho_i(x)\big) \to \rho_j(x) \Big).$$

So, we have

$$\rho_{\omega_2}^{\mathcal{M}}(a) < ... < \rho_2^{\mathcal{M}}(a) < \rho_1^{\mathcal{M}}(a) < R^{\mathcal{M}}(a) < 1,$$

a contradiction with cardinality of $[0,1]$. *But, one can easily verify that* T *is finitely satisfiable.*

We show that if ones consider the metric

$$d_{max}(x,y) = \begin{cases} \max\{x,y\} & x \neq y, \\ 0 & x = y. \end{cases}$$

on the set of truth values, then the truth functionality of all logical connectives of Gödel logic become continuous. This fact leads us to use the ultraproduct method to prove the compactness theorem for first-order Gödel logic

2. Main Results

Let V be a Gödel set, i. e., a closed subset of $[0,1]$ with the Euclidean topology which is contained both 0 and 1. For a given first-order language $\mathcal{L}$, the concept of $\mathcal{L}$-structure is defined as usual. The metrically semantic of first-order Gödel logic $\mathfrak{G}_V$ is defined as follows.

Definition 2.1. *Let* $\mathcal{M}$ *be an* $\mathcal{L}$*-structure in the first-order Gödel logic* $\mathfrak{G}_V$. *For an* n*-tuple* $\bar{x}$, *the interpretation of* $\mathcal{L}$*-formula* $\varphi(\bar{x})$ *is a function* $\varphi^{\mathcal{M}} : M^n \to V$ *which is inductively defined as follows:*

- $\bot^{\mathcal{M}} = 1$.
- *For every* n*-ary predicate symbol* P,
$$P(t_1,...,t_n)^{\mathcal{M}}(\bar{a}) = P^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}),...,t_n^{\mathcal{M}}(\bar{a})).$$
- $(\varphi \wedge \psi)^{\mathcal{M}}(\bar{a}) = \max\{\varphi^{\mathcal{M}}(\bar{a}), \psi^{\mathcal{M}}(\bar{a})\}$.
- $(\varphi \to \psi)^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{M}}(\bar{a}) \dot{\to} \psi^{\mathcal{M}}(\bar{a})$, *where* $x \dot{\to} y = \begin{cases} 0 & x \geq y, \\ y & x < y. \end{cases}$
- *If* $\varphi(\bar{x}) = \forall y\ \psi(y,\bar{x})$ *then* $\varphi^{\mathcal{M}}(\bar{a}) = \sup\limits_{b \in M}\{\psi^{\mathcal{M}}(b,\bar{a})\}$.
- *If* $\varphi(\bar{x}) = \exists y\ \psi(y,\bar{x})$ *then* $\varphi^{\mathcal{M}}(\bar{a}) = \inf\limits_{b \in M}\{\psi^{\mathcal{M}}(b,\bar{a})\}$.

An easy argument show that

- $(\varphi \vee \psi)^{\mathcal{M}}(\bar{a}) = \min\{\varphi^{\mathcal{M}}(\bar{a}), \psi^{\mathcal{M}}(\bar{a})\}$.
- $(\varphi \leftrightarrow \psi)^{\mathcal{M}}(\bar{a}) = d_{max}\big(\varphi^{\mathcal{M}}(\bar{a}), \psi^{\mathcal{M}}(\bar{a})\big)$.

Note that d_{max} is an ultrametric on $[0,1]$. Furthermore, it is induces an ultrametric d_{Max} on $[0,1]^2$ which is defined by

$$d_{Max}\big((x,y),(x_1,y_1)\big) = \max\big\{d_{max}(x,x_1), d_{max}(y,y_1)\big\}.$$

Lemma 2.2. *The functions* $\max$, $\min$, *and* $\dot{\rightarrow}$ *are continuous functions from* $([0,1]^2, d_{Max})$ *into* $([0,1], d_{max})$.

(Fact1) A filter $\mathfrak{D}$ on a topological space X, convergent to an element $x \in X$, whenever for each open set U containing x, U is an element of $\mathfrak{D}$. This is denoted by $\mathfrak{D} \to x$ and x is called a limit point of $\mathfrak{D}$.

(Fact2) X is a compact Hausdorff space if and only if every filter $\mathcal{D}$ on X has a unique limit point.

(Fact3) Let $f : X \to Y$ be a continuous function at $x_0 \in X$ and $\mathfrak{D}$ be a filter on X. If $f(\mathfrak{D})$ be the filter on Y generated by the set $\{f(A) : A \in \mathfrak{D}\}$, then $f(\mathfrak{D}) \to f(x_0)$.

Definition 2.3. *Let* X *be a topological space,* I *be a nonempty set, and* $\mathfrak{D}$ *be a filter on* I. *Furthermore, let* $f \in I^X$, $\{x_i\}_{i\in I}$ *be the range of* f, *and* $f^*(\mathfrak{D}) = \{A \subseteq X : f^{-1}(A) \in \mathfrak{D}\}$. *If* $f^*(\mathfrak{D})$ *is convergent to* $x \in X$, *then we call* x *the* $\mathfrak{D}$*-limit of the family* $\{x_i\}_{i\in I}$ *and write* $\lim_{\mathfrak{D}} x_i = x$.

Another version of (Fact3), is the following.

Corollary 2.4. *Let* $f : X \to Y$ *be a continuous function at* $x_0 \in X$, I *be a nonempty set, and* $\mathfrak{D}$ *be a filter on* I. *If* $\lim_{\mathfrak{D}} x_i = x_0$ *then* $\lim_{\mathfrak{D}} f(x_i) = f(x_0)$.

Lemma 2.5. *Let* V *be a Gödel set,* I *be a nonempty set, and* $\mathfrak{D}$ *be a filter on* I. *Consider* (V, d_{max}) *as a topological space. If* $\{x_i\}_{i\in I}$ *and* $\{y_i\}_{i\in I}$ *are two family of elements of* V, *then*

$$\lim_{\mathfrak{D}} x_i \leq \lim_{\mathfrak{D}} y_i \text{ if and only if } \{i : x_i \leq y_i\} \in \mathfrak{D}.$$

Now, assume that V is a Gödel set which is a compact Hausdorff subspace of $([0,1], d_{max})$. For example assume that

$$V = V_{\downarrow} = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}.$$

Then by (Fact2), we could construct the ultraproduct of a family of structures in the first-order Gödel logic $\mathfrak{G}_V$.

Definition 2.6. *Let* $\{\mathcal{M}_i\}_{i\in I}$ *be a family of* $\mathcal{L}$*-structures and* $\mathfrak{D}$ *be a filter on* I. *The* $\mathfrak{D}$*-ultraproduct of family* $\{\mathcal{M}_i\}_{i\in I}$ *is an* $\mathcal{L}$*-structure* $\mathcal{M}$ *with universe* $M = \prod_{i\in I} M_i$ *whose interpretation of elements of* $\mathcal{L}$ *is defined as follows.*

- *For n-ary predicate symbol $R \in \mathcal{L}$, $R^{\mathcal{M}} : M^n \to V$ is defined by*
$$R^{\mathcal{M}}\big(\{x_i^1\}_{i\in I}, ..., \{x_i^n\}_{i\in I}\big) = \lim\nolimits_{\mathfrak{D}} R^{\mathcal{M}_i}(x_i^1, ..., x_i^n).$$
- *For n-ary function symbol $f \in \mathcal{L}$, $f^{\mathcal{M}} : M^n \to M$ is defined by*
$$f^{\mathcal{M}}\big(\{x_i^1\}_{i\in I}, ..., \{x_i^n\}_{i\in I}\big) = \{f^{\mathcal{M}_i}(x_i^1, ..., x_i^n)\}_{i\in I}.$$

Obviously, by (Fact2) the above definition is well-defined.

Theorem 2.7. *(Łoś theorem) Let V be a Gödel set and (V, d_{max}) be a compact Hausdorff space. Furthermore, assume that $\{\mathcal{M}_i\}_{i\in I}$ be a family of $\mathcal{L}$-structures. If $\mathfrak{D}$ is an ultrafilter on I and $\mathcal{M}$ is the $\mathfrak{D}$-ultraproduct of family $\{\mathcal{M}_i\}_{i\in I}$, then in first-order Gödel logic $\mathfrak{G}_V$, for each $\mathcal{L}$-formula $\varphi(x_1, ..., x_n)$ and each $\mathbf{a}_k = \{a_i^k\}_{i\in I} \in M$ $(1 \leq k \leq n)$,*
$$\varphi^{\mathcal{M}}(\mathbf{a}_1, ..., \mathbf{a}_n) = \lim_{\mathfrak{D}} \varphi^{\mathcal{M}_i}(a_i^1, ..., a_i^n).$$

Proof. Using Lemma 2.2 the proof is as like as [2]. □

Theorem 2.8. *(Compactness theorem) Let V be a Gödel set and (V, d_{max}) be a compact Hausdorff space. In first-order Gödel logic $\mathfrak{G}_V$, every finitely satisfiable theory is satisfiable.*

Proof. Assume that T is a finitely satisfiable theory. Let I be the set of all finite subsets of T. For each $\varphi \in T$, let $\overline{\varphi} = \{\Sigma : \varphi \in \Sigma \text{ and } \Sigma \in I\}$. Obviously $\mathfrak{T} = \{\overline{\varphi} : \varphi \in T\}$ has the finite intersection property. So, there exists an ultrafilter $\mathfrak{D}$ on I containing $\mathfrak{T}$.

Let $T_i \in I$. As T is finitely satisfiable, there exists a structure $\mathcal{M}_i \models T_i$. Suppose that $\mathcal{M}$ be the $\mathfrak{D}$-ultraproduct of $\{\mathcal{M}_i\}_{i\in I}$. By Łoś theorem, $\mathcal{M} \models T$. □

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