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ADDITIVE GÖDEL LOGIC: AN ENRICHED GÖDEL LOGIC WITH ŁUKASIEWICZ CONNECTIVES

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ABSTRACT. We further develop ideas of rational Gödel logic to study extensions of first-order Gödel logic, called Additive Gödel logic. A relevant model theory is developed for this logic to show that it enjoys some nice properties such as Robinson joint consistency theorem. Moreover, it is shown that the class of (ultrametric exhaustive) models with respect to elementary substructure forms an abstract elementary class. This is a joint work with Massoud Pourmahdian†.

1. INTRODUCTION

Extending model-theoretic techniques from classical model theory to other logics is a fashionable trend. The merit of this trend is twofold. Firstly, it can be viewed as a measurement for complexity of semantical aspects of a given logic and, secondly, can be used as an instrumental tool to verify certain fundamental logical questions. Following this, the present paper can been as further development initiated in [1] for studying model-theoretic aspects of extensions of first-order Gödel Logic. While in [1], the first-order Gödel Logic is enriched by adding countably many nullary logical constants for rational numbers, here we extend it in other way by adding a group structure on the set of truth values. This extension enables us to strengthen considerably the expressive power of the Gödel Logic. On the other hand, we will see that this strengthening does not prevent us to have nice model-theoretic properties. Therefore, this extension enjoys a balance between the expressive power, on one hand, and nice model-theoretic properties, on the other hand.

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The additive Gödel logic not only involve the Gödel logic but also it includes the Łukasiewicz logic. So, this logic can be viewed as a common non-trivial extension of both of Gödel and Łukasiewicz logic.

2. Additive Gödel Logic

Definition 2.1. *The first-order additive Gödel logic, $AG\forall$, consists of the following logical symbols:*

1. *Logical connectives* $\wedge$, $\rightarrow$, $\otimes$, $^{-1}$, $\bar{1}$ *and* $\perp$.
2. *Quantifiers* $\forall$ *and* $\exists$.
3. *A countable set of variables* $\{x_n\}_{n\in\mathbb{N}}$.

Further connectives like as $\vee$ and $\neg$ are defined as usual. The others could be defined as follows.

$$\begin{aligned}
\varphi^n &:= \varphi^{n-1}\otimes\varphi \\
\varphi\prec\psi &:= (\psi\rightarrow\varphi)\rightarrow\psi \\
\varphi\ll\psi &:= ((\varphi\prec\psi)\wedge\neg\neg\psi^{-1})\vee(\psi\wedge\neg\neg\varphi^{-1}) \\
\Delta(\varphi) &:= \neg(\varphi\ll T) \\
\varphi\rightarrow_L\psi &:= \bar{1}\rightarrow(\psi\otimes\varphi^{-1})
\end{aligned}$$

The semantic of additive Gödel logic is based on totally ordered Abelian groups.

Definition 2.2. *Let $(G,*,\leq)$ be a totally ordered Abelian group with an identity element 1_G. Set $\Gamma_G=\{0\}\cup G\cup\{\infty\}$, and let*

$\infty*0=0*\infty=1_G$,
for all $a\in G$, $a*\infty=\infty*a=\infty$ and $a*0=0*a=0$,
$0^{-1}=\infty$ *and* $\infty^{-1}=0$.

Extend the order $\leq$ on Γ_G such that 0 and ∞ be the least and largest elements of Γ_G.

For a given totally ordered Abelian group G, we consider Γ_G as the set of truth values, whereas 0 is the absolute falsity and ∞ is the absolute truth.

For a given language τ, a *τ-structure* $\mathcal{M}$ is a nonempty set M called the universe of $\mathcal{M}$ together with a totally ordered Abelian group $(G,*,\leq)$ in which the interpretation of predicate and function symbols are functions such as $P^{\mathcal{M}}:M^n\rightarrow\Gamma_G$ and $f^{\mathcal{M}}:M^n\rightarrow M$. In this way, we call $\mathcal{M}$ a τ^G-structure and denote $\mathcal{M}$ by $\mathcal{M}=(G,M)$.

The interpretation of a formula $\varphi(\bar{x})$ in a τ^G-structure $\mathcal{M}$ is a function $\varphi^{\mathcal{M}}:M^n\rightarrow\Gamma_G$ which is inductively determined as follows.

1. $\perp^{\mathcal{M}}=0$, $\top^{\mathcal{M}}=\infty$ and $\bar{1}=1_G$.
2. For every n-ary predicate symbol P,

$$P^{\mathcal{M}}(t_1(\bar{a}),...,t_n(\bar{a})) = P^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}),...,t_n^{\mathcal{M}}(\bar{a})).$$

(3) $(\varphi \wedge \psi)^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{M}}(\bar{a}) \curlywedge \psi^{\mathcal{M}}(\bar{a})$.
(4) $(\varphi \to \psi)^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{M}}(\bar{a}) \dot{\to} \psi^{\mathcal{M}}(\bar{a})$.
(5) $(\varphi \otimes \psi)^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{M}}(\bar{a}) * \psi^{\mathcal{M}}(\bar{a})$.
(6) $(\varphi^{-1})^{\mathcal{M}}(\bar{a}) = (\varphi^{\mathcal{M}}(\bar{a}))^{-1}$.
(7) if $\varphi(\bar{x}) = \forall y\ \psi(y,\bar{x})$ then $\varphi^{\mathcal{M}}(\bar{a}) = \inf_{b\in M}\{\psi^{\mathcal{M}}(b,\bar{a})\}$.
(8) if $\varphi(\bar{x}) = \exists y\ \psi(y,\bar{x})$ then $\varphi^{\mathcal{M}}(\bar{a}) = \sup_{b\in M}\{\psi^{\mathcal{M}}(b,\bar{a})\}$.

We use the notion of finite entailment instead of the notion of proof. For a theory T and a sentence φ, $T \overset{f}{\models} \varphi$ if there exist a finite subset S of T such that $S \models \varphi$. However, note that the concept of proof could be developed also.

Using the Henkin construction, we obtain a version of compactness theorem for additive Gödel logic.

Theorem 2.3. *Let T be a τ-theory and χ be a τ-sentence. $T \models \chi$ if and only if $T \overset{f}{\models} \chi$.*

Corollary 2.4. *(Compactness Theorem) A theory T is satisfiable if and only if it is finitely satisfiable.*

Remark 2.5. *If ϵ and ρ be two nullary predicate symbols, then $T = \{\bar{1} \ll \rho, \epsilon \ll \top\} \cup \{\rho^n \ll \epsilon\}_{n\in\mathbb{N}}$ is finitely satisfiable by standard models, but it has no standard model. Note that an easy calculation show that for any two sentences φ, ψ and any structure $\mathcal{M}$, $\mathcal{M} \models \varphi \ll \psi$ if and only if $\varphi^{\mathcal{M}} < \psi^{\mathcal{M}}$.*

3. Some Model Theory

Fix a first-order language τ_e including a binary predicate symbol e. This predicate plays the same role as the equality relation in classical first-order logic. In This case the interpretation of e^{-1} in a τ_e-structure $\mathcal{M}$ is as like as a pseudo-ultrametric on the universe of $\mathcal{M}$.

Definition 3.1. *Let $\mathcal{M} = (G, M)$ be a τ_e-structure. We call $\mathcal{M}$ An ultrametric structure, whenever for all $a, b, c \in M$*

- $(e^{-1})^{\mathcal{M}}(a,b) = 0$ *if and only if* $a = b$,
- $(e^{-1})^{\mathcal{M}}(a,b) = (e^{-1})^{\mathcal{M}}(b,a)$,
- $(e^{-1})^{\mathcal{M}}(a,b) \le \max\{(e^{-1})^{\mathcal{M}}(a,c), (e^{-1})^{\mathcal{M}}(b,c)\}$.

Theorem 3.2. *(Compactness Theorem) A theory T is finitely satisfiable by ultrametric models if and only if it is satisfiable by an ultrametric model.*

Definition 3.3. *For a τ-structure $\mathcal{M} = (G, M)$ let $Gr(\mathcal{M})$ be the ordered subgroup of truth values of all τ-formulas, i.e.,*

$$Gr(\mathcal{M}) = \{\varphi^{\mathcal{M}}(\bar{a}) : \varphi \in Form(\tau), \bar{a} \subseteq M\} \setminus \{0, \infty\}.$$

$\mathcal{M} = (G, M)$ is called an exhaustive structure if $G = Gr(\mathcal{M})$.

Definition 3.4. *Let $\mathcal{M} = (G, M)$ and $\mathcal{N} = (H, N)$ be τ-structures. We say that $\mathcal{M}$ is elementary embedded in $\mathcal{N}$, if there are an injection $h : M \to N$ and a strict order preserving group homeomorphism $T : G \to H$ such that $h(f^{\mathcal{M}}(a_1, ..., a_{n_f})) = f^{\mathcal{N}}(h(a_1), ..., h(a_{n_f}))$ and $T\big(\varphi^{\mathcal{M}}(b_1, b_2, ..., b_n)\big) = \varphi^{\mathcal{N}}(h(b_1), h(b_2), ..., h(b_n))$ for all function symbol $f \in \tau$, $\bar{a} \in M^{n_f}$, τ-formula φ and $\bar{b} \subseteq M$. In this case, we call $(h, T) : \mathcal{M} \hookrightarrow_{\tau} \mathcal{N}$ an elementary embedding from $\mathcal{M}$ into $\mathcal{N}$.*

Remark 3.5. *If $\mathcal{M}$ and $\mathcal{N}$ be exhaustive τ structures and $h : M \to N$ be an injection such that preserve all τ-formulas and $I_{\mathcal{MN}} : Gr(\mathcal{M}) \to Gr(\mathcal{N})$ is defined by $I_{\mathcal{MN}}\big(\varphi^{\mathcal{M}}(a_1, ..., a_n)\big) = \varphi^{\mathcal{N}}(h(a_1), ..., h(a_n))$, then $(h, I_{\mathcal{MN}})$ is an elementary embedding from $\mathcal{M}$ into $\mathcal{N}$.*

Theorem 3.6. *(Amalgamation over ultrametric structures) Let $\mathcal{A} = (G_A, A)$, $\mathcal{B} = (G_B, B)$ and $\mathcal{M} = (G_M, M)$ be three exhaustive ultrametric τ_e-structures. Suppose also, $(j, I_{\mathcal{MA}}) : \mathcal{M} \hookrightarrow_{\tau_e} \mathcal{A}$ and $(k, I_{\mathcal{MB}}) : \mathcal{M} \hookrightarrow_{\tau_e} \mathcal{B}$ are elementary embeddings. Then, there are exhaustive ultrametric τ_e-structure $\mathcal{N} = (G_N, N)$ and elementary embeddings $(j_1, I_{\mathcal{AN}}) : \mathcal{A} \hookrightarrow_{\tau_e} \mathcal{N}$ and $(k_1, I_{\mathcal{BN}}) : \mathcal{B} \hookrightarrow_{\tau_e} \mathcal{N}$ such that $j_1 \circ j = k_1 \circ k$.*

Theorem 3.7. *(Union of chain) Let $\{\mathcal{M}_i\}_{i=0}^{\infty}$ be a sequence of exhaustive τ-structures such that $\mathcal{M}_i \prec \mathcal{M}_{i+1}$ for each $i \geq 1$. There exists a unique τ-structure $\mathcal{M}$ with the underlying universe $M = \cup_{i=1}^{\infty} M_i$ such that $\mathcal{M}_i \prec \mathcal{M}$ for each $i \geq 1$.*

Theorem 3.8. *(Downward Löwenheim-Skolem) Let $\mathcal{M}$ be a τ-structures and $A \subseteq M$. There is an elementary substructure $\mathcal{N}$ of $\mathcal{M}$ such that $A \subseteq N$ and $|N| \leq |A| + \|\tau\|$.*

Theorem 3.9. *(Robinson consistency theorem) Suppose $\{T_i\}_{i=1,2}$ be satisfiable τ_e^i-theories by ultrametric models. Suppose also $\tau_\cap = \tau_e^1 \cap \tau_e^2$ and $\tau_\cup = \tau_e^1 \cup \tau_e^2$. If $T = T_1 \cap T_2$ is a linear complete $\tau_\cap$-theory, then the $\tau_\cup$-theory $T_1 \cup T_2$ has an ultrametric model.*

References

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