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On semi-convergence of the improved symmetric successive over-relaxation method for singular saddle point problems[1]

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Abstract

In this paper, we analyze the semi-convergence of the improved symmetric successive over-relaxation method for singular saddle point problems. In fact, when the $(1, 2)$-block of the coefficient matrix is rank deficient, the saddle point problem is singular. Here, we study sufficient conditions for semi-convergence of the improved symmetric successive over-relaxation method.

Keywords: Saddle point problem, Iterative method, Semi-convergence, Improved SSOR, Preconditioner

Mathematics Subject Classification [2010]: 65F08, 65F10

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1 Introduction

Consider the following large and sparse saddle point problem

$$\mathcal{A}u \equiv \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix} \equiv b, \tag{1}$$

where $A \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix, $B \in \mathbb{R}^{m\times n}$ is a rank deficient matrix, $p \in \mathbb{R}^m$, and $q \in \mathbb{R}^n$ with $n \leq m$. It follows that the coefficient matrix of the saddle point problem (1) is singular.

The saddle point problems is crucially important in a variety of scientific and engineering applications [3]. Here we mention some applications of the saddle point problems like mixed finite element approximation of elliptic partial differential equations, optimal control, computational fluid dynamics, weighted least-squares problems, electronic networks, computer graphics and nonlinearly constrained optimization, and so forth [1, 2, 5].

When B in (1) is of full rank, a number of iteration methods and their numerical properties have been discussed to solve the saddle point problem (1) in the literature, such as SOR-like method presented in [1] and improved SSOR method given in [2].

Though most often the matrix B occur in the form of full column rank, but not always in practise. For example, in the finite difference discretization of the Navier-Stokes equation with periodic boundary conditions, B in (1) becomes singular [5]. In recent years,

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[1] Dedicated to Alireza Afzalipour and Fakhereh Saba, the founders of Kerman University

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there has been a surge of interest in solving singular saddle point problems (1), which some of them are Uzawa-type method proposed in [5] and Uzawa-HSS method given in [4].

In this paper, the idea of the improved symmetric successive over-relaxation method is established for the singular saddle point problems (ISSORS) of the form (1). We present the semi-convergence conditions of the proposed ISSORS method. Finally, some conclusions are presented.

2 Improved SSOR iteration method

In this section, we proposed improved symmetric successive over-relaxation method given in [2] for solving saddle point problem (1). So, we consider the following splitting

$$\mathcal{A} = \mathcal{D} - \mathcal{A}_l - \mathcal{A}_u, \tag{2}$$

where

$$\mathcal{D} = \begin{pmatrix} A & 0 \\ 0 & Q \end{pmatrix}, \quad \mathcal{A}_l = \begin{pmatrix} -\frac{1}{2}A & 0 \\ B^T & \frac{1}{2}Q \end{pmatrix}, \quad \mathcal{A}_u = \begin{pmatrix} \frac{1}{2}A & -B \\ 0 & \frac{1}{2}Q \end{pmatrix},$$

for nonsingular and symmetric matrix $Q \in \mathbb{R}^{n\times n}$. Set

$$\mathcal{L} = \mathcal{D}^{-1}\mathcal{A}_l = \begin{pmatrix} -\frac{1}{2}I & 0 \\ Q^{-1}B^T & \frac{1}{2}I \end{pmatrix}, \quad \mathcal{U} = \mathcal{D}^{-1}\mathcal{A}_u = \begin{pmatrix} \frac{1}{2}I & -A^{-1}B \\ 0 & \frac{1}{2}I \end{pmatrix}, \tag{3}$$

where I is the identity matrix of the appropriate dimension.

Suppose that $u^{(k)} = [x^{(k)T}, y^{(k)T}]^T$ is the k-th approximation of the exact solution of (1), then improved symmetric successive over-relaxation iterative method is obtained as

$$u^{(k+\frac{1}{2})} = (I-\omega\mathcal{L})^{-1}((1-\omega)I - \omega\mathcal{U})u^{(k)} + \omega(I-\omega\mathcal{L})^{-1}\mathcal{D}^{-1}b, \tag{4}$$

$$u^{(k+1)} = (I-\omega\mathcal{U})^{-1}((1-\omega)I - \omega\mathcal{L})u^{(k+\frac{1}{2})} + \omega(I-\omega\mathcal{U})^{-1}\mathcal{D}^{-1}b. \tag{5}$$

So, based on Eqs. (4) and (5) and assuming $\omega \neq \pm 2$, the ISSORS method can be considered as the following algorithm.

Algorithm 2.1. Improved SSOR iteration method

Given initial guess $u^{(0)} = [(x^{(0)})^T, (y^{(0)})^T]^T$. For $k = 0, 1, 2, \ldots$ until iteration sequence $\{[(x^{(k)})^T, (y^{(k)})^T]^T\}$ is convergent, compute

1. $y^{(k+1)} = y^{(k)} + \frac{4\omega}{2+\omega}Q^{-1}B^T\left\{x^{(k)} + \frac{2\omega}{2-\omega}A^{-1}(p - By^{(k)})\right\} - \frac{4\omega}{2-\omega}Q^{-1}q.$
2. $x^{(k+1)} = \frac{2-3\omega}{2+\omega}x^{(k)} - \frac{2\omega}{2-\omega}A^{-1}B\left\{y^{(k+1)} + \frac{2-3\omega}{2+\omega}y^{(k)}\right\} + \frac{4\omega}{2+\omega}A^{-1}p.$

3 Semi-convergence of the ISSORS

In this section, we discuss the semi-convergence of the ISSORS for solving singular saddle point problem (1). The following Lemma provides the necessary and sufficient conditions for semi-convergence of a stationary iterative method [4, 5].

Lemma 3.1. *Let G be a nonsingular matrix. Then, iteration scheme $x^{(k+1)} = x^{(k)} + G(d - Mx^{(k)})$ used for solving singular linear system $Mx = d$ is semi-convergent if and only if the following two conditions are fulfilled* *(i)* $\operatorname{index}(I-T)=1$, *or equivalently,* $\operatorname{rank}(I-T)^2=\operatorname{rank}(I-T)$, *where* $T=I-GM$ *is the iteration matrix.*

(ii) The pseudo-spectral radius of T *is less than 1, i.e.*

$$\nu(T)=\max\{|\lambda|:\lambda\in\sigma(T)\ \ \textit{and}\ \ \lambda\neq 1\}<1,$$

where $\sigma(T)$ *is the spectrum of the matrix* T.

Here we denote the null space of the matrix A by $\operatorname{null}(A)$. To analyze the semi-convergence properties of the ISSORS iteration method for the singular saddle point problem (1), we write the Algorithm 2.1 as

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix}=\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix}+\mathcal{G}\left(b-\mathcal{A}\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix}\right), \tag{6}$$

where

$$\begin{aligned}\mathcal{G}&=\omega(2-\omega)(I-\omega\mathcal{U})^{-1}(I-\omega\mathcal{L})^{-1}\mathcal{D}^{-1}\\ &=\begin{pmatrix} \frac{4\omega}{2+\omega}A^{-1}-\frac{16\omega^2}{(2-\omega)(4-\omega^2)}A^{-1}BQ^{-1}B^TA^{-1} & -\frac{8\omega^2}{(2-\omega)^2}A^{-1}BQ^{-1} \\ \frac{8\omega^2}{4-\omega^2}Q^{-1}B^TA^{-1} & \frac{4\omega}{2-\omega}Q^{-1}\end{pmatrix}.\end{aligned} \tag{7}$$

By what mentioned above, we can obtain the iteration matrix of the scheme (6) as

$$\begin{aligned}\mathcal{T}&=I-\mathcal{G}\mathcal{A}\\ &=\begin{pmatrix} \frac{2-3\omega}{2+\omega}I-\frac{8\omega^2}{4-\omega^2}A^{-1}BQ^{-1}B^T & -\frac{4\omega}{2+\omega}A^{-1}B+\frac{16\omega^2}{(2-\omega)(4-\omega^2)}A^{-1}BQ^{-1}B^TA^{-1}B \\ \frac{4\omega}{2+\omega}Q^{-1}B^T & I-\frac{8\omega^2}{4-\omega^2}Q^{-1}B^TA^{-1}B\end{pmatrix}.\end{aligned} \tag{8}$$

Now, we prove that the ISSORS iteration method is semi-convergent for solving the singular saddle point problem (1) under some restrictions as in Lemma 3.1. It is necessary to mention that, in this algorithm, the matrix Q is an approximation of the Schur complement matrix $B^TA^{-1}B$ [1,2].

Lemma 3.2. *Let* $A\in\mathbb{R}^{m\times m}$ *be symmetric positive definite,* $B\in\mathbb{R}^{m\times n}$ *be rank deficient, and* $Q\in\mathbb{R}^{n\times n}$ *be nonsingular and symmetric. Then iteration matrix* $\mathcal{T}$ *of the ISSORS method given in Eq. (8) satisfies*

$$\operatorname{index}(I-\mathcal{T})=1. \tag{9}$$

Proof. Inasmuch as $\mathcal{T}=I-\mathcal{G}\mathcal{A}$ in (8), so Eq. (9) holds if

$$\operatorname{null}(\mathcal{G}\mathcal{A})=\operatorname{null}((\mathcal{G}\mathcal{A})^2),$$

where $\mathcal{G}$ is defined as in (7). It is obvious that $\operatorname{null}(\mathcal{G}\mathcal{A})\subseteq\operatorname{null}((\mathcal{G}\mathcal{A})^2)$.

Let $x=[x_1^T,x_2^T]^T\in\mathbb{R}^{m+n}$ satisfies $(\mathcal{G}\mathcal{A})^2x=0$. Denote $y=(\mathcal{G}\mathcal{A})x$. So, we have

$$y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=\begin{pmatrix} \frac{4\omega}{2+\omega}A^{-1}-\frac{16\omega^2}{(2-\omega)(4-\omega^2)}A^{-1}BQ^{-1}B^TA^{-1} & -\frac{8\omega^2}{(2-\omega)^2}A^{-1}BQ^{-1} \\ \frac{8\omega^2}{4-\omega^2}Q^{-1}B^TA^{-1} & \frac{4\omega}{2-\omega}Q^{-1}\end{pmatrix}\times \tag{10}$$

$$\begin{pmatrix} A & B \\ -B^T & 0\end{pmatrix}\begin{pmatrix} x_1 \\ x_2\end{pmatrix}. \tag{11}$$

This results in

$$\begin{cases} y_1 = \left(\frac{4\omega}{2+\omega}I - \frac{8\omega^3}{(2-\omega)(4-\omega^2)}A^{-1}BQ^{-1}B^T\right)x_1 \\ \qquad + \left(\frac{4\omega}{2+\omega}A^{-1}B - \frac{16\omega^2}{(2-\omega)(4-\omega^2)}A^{-1}BQ^{-1}B^TA^{-1}B\right)x_2, \\ y_2 = -\frac{4\omega}{2+\omega}Q^{-1}B^Tx_1 + \frac{8\omega^2}{4-\omega^2}Q^{-1}B^TA^{-1}Bx_2. \end{cases} \tag{12}$$

Since $\mathcal{G}$ is invertible and $(\mathcal{G}\mathcal{A})y = (\mathcal{G}\mathcal{A})^2x = 0$, it holds that $\mathcal{A}y = 0$, i.e.,

$$Ay_1 + By_2 = 0, \quad -B^Ty_1 = 0. \tag{13}$$

It is easy to get $y_1 = -A^{-1}By_2$. Then we obtain $B^TA^{-1}By_2 = 0$. Therefore $By_2 = 0$ and $y_1 = 0$. From $By_2 = 0$, we attain $x_1 = \frac{2\omega}{2-\omega}A^{-1}Bx_2$, that means $y_2 = 0$. Thus $y = (\mathcal{G}\mathcal{A})x = 0$, i.e.,

$$\mathrm{null}((\mathcal{G}\mathcal{A})^2) \subseteq \mathrm{null}(\mathcal{G}\mathcal{A}).$$

The proof is complete. □

Let the singular value decomposition of matrix B be as

$$B = U(B_r, 0)V^T, \quad B_r = \begin{pmatrix} \Sigma_r \\ 0 \end{pmatrix} \in \mathbb{R}^{m\times r}, \quad \Sigma_r = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r) \in \mathbb{R}^{r\times r}, \tag{14}$$

with $U \in \mathbb{R}^{m\times m}, V \in \mathbb{R}^{n\times n}$ being two orthogonal matrices and $\sigma_i (i = 1, 2, \ldots, r)$ being singular values of B. We define

$$P = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}.$$

It is obvious that P is a $(m + n) \times (m + n)$ orthogonal matrix, and therefore we have the orthogonal similarities

$$\hat{\mathcal{T}} = P^T\mathcal{T}P, \quad \hat{A} = U^TAU, \quad \hat{Q} = V^TQV. \tag{15}$$

Hence $\mathcal{T}$ has the same spectrum as the matrix $\hat{\mathcal{T}}$. Let $[\hat{Q}^{-1}]_{i:j,k:s}$ stands for the submatrix of $\hat{Q}^{-1}$ by considering rows i to j and columns from k to s(we use the notation $[\cdot]_r$ for the case $[\cdot]_{1:r,1:r}$),

Lemma 3.3. *Suppose $A \in \mathbb{R}^{m\times m}$ be symmetric positive definite, $B \in \mathbb{R}^{m\times n}$ be rank deficient. Let $\hat{A}$ and $\hat{Q}$ are defined as in (15). Assume that for B_r defined in (14), all eigenvalues of $\hat{Q}^{-1}B_r^T\hat{A}^{-1}B_r$ are real and positive. Then, pseudo-spectral of the iteration matrix $\mathcal{T}$ of the ISSORS method is less that one if*

$$0 < \omega < \frac{2}{1 + 2\sqrt{\mu}}, \tag{16}$$

where $\mu = \rho(\hat{Q}^{-1}B_r^T\hat{A}^{-1}B_r)$.

Proof. Firstly, by definition of $\hat{\mathcal{T}}$ in (15), it holds that

$$\hat{\mathcal{T}} = \begin{pmatrix} \hat{\mathcal{T}}_{11} & \hat{\mathcal{T}}_{12} \\ \hat{\mathcal{T}}_{21} & \hat{\mathcal{T}}_{22} \end{pmatrix}, \tag{17}$$

where

$$\hat{\mathcal{T}}_{11} = \frac{2 - 3\omega}{2 + \omega}I_m - \frac{8\omega^2}{(4 - \omega^2)}U^TA^{-1}BQ^{-1}B^TU$$

$$= \frac{2-3\omega}{2+\omega} I_m - \frac{8\omega^2}{(4-\omega^2)} \hat{A}^{-1}(B_r, 0)\hat{Q}^{-1} \begin{pmatrix} B_r^T \\ 0 \end{pmatrix}$$
$$= \frac{2-3\omega}{2+\omega} I_m - \frac{8\omega^2}{(4-\omega^2)} \hat{A}^{-1} B_r [\hat{Q}^{-1}]_r B_r^T, \tag{18}$$

$$\begin{aligned}\hat{\mathcal{T}}_{12} &= -\frac{4\omega}{2+\omega} U^T A^{-1} B V + \frac{16\omega^2}{(2-\omega)(4-\omega^2)} U^T A^{-1} B Q^{-1} B^T A^{-1} B V \\ &= -\frac{4\omega}{2+\omega} \hat{A}^{-1}(B_r, 0) + \frac{16\omega^2}{(2-\omega)(4-\omega^2)} \hat{A}^{-1}(B_r, 0)\hat{Q}^{-1} \begin{pmatrix} B_r^T \\ 0 \end{pmatrix} \hat{A}^{-1}(B_r, 0) \\ &= \left(-\frac{4\omega}{2+\omega} \hat{A}^{-1} B_r + \frac{16\omega^2}{(2-\omega)(4-\omega^2)} \hat{A}^{-1} B_r [\hat{Q}^{-1}]_r B_r^T \hat{A}^{-1} B_r, 0 \right), \end{aligned} \tag{19}$$

$$\begin{aligned}\hat{\mathcal{T}}_{21} &= \frac{4\omega}{2+\omega} V^T Q^{-1} B^T U \\ &= \frac{4\omega}{2+\omega} \hat{Q}^{-1} \begin{pmatrix} B_r^T \\ 0 \end{pmatrix} \\ &= \frac{4\omega}{2+\omega} \begin{pmatrix} [\hat{Q}^{-1}]_r B_r^T \\ [\ \hat{Q}^{-1}]_{r+1,n,1:r}\ \mathrm{B}_r^T \end{pmatrix}, \end{aligned} \tag{20}$$

$$\begin{aligned}\hat{\mathcal{T}}_{22} &= I_n - \frac{8\omega^2}{4-\omega^2} V^T Q^{-1} B^T A^{-1} B V \\ &= I_n - \frac{8\omega^2}{4-\omega^2} \hat{Q}^{-1} \begin{pmatrix} B_r^T \\ 0 \end{pmatrix} \hat{A}^{-1}(B_r, 0) \\ &= \begin{pmatrix} I_r - \frac{8\omega^2}{4-\omega^2} [\hat{Q}^{-1}]_r B_r^T \hat{A}^{-1} B_r & 0 \\ 0 & I_{n-r} \end{pmatrix}. \end{aligned} \tag{21}$$

It follows from (18)-(21) that

$$\hat{\mathcal{T}} = \begin{pmatrix} \tilde{\mathcal{T}} & 0 \\ \begin{pmatrix} 0 \\ [\ \hat{Q}^{-1}]_{r+1,n,1:r}\ \mathrm{B}_r^T \end{pmatrix} & I_{n-r} \end{pmatrix}, \tag{22}$$

with

$$\tilde{\mathcal{T}} = \begin{pmatrix} \frac{2-3\omega}{2+\omega} I_m - \frac{8\omega^2}{(4-\omega^2)} \hat{A}^{-1} B_r [\hat{Q}^{-1}]_r B_r^T & -\frac{4\omega}{2+\omega} \hat{A}^{-1} B_r + \frac{16\omega^2}{(2-\omega)(4-\omega^2)} \hat{A}^{-1} B_r [\hat{Q}^{-1}]_r B_r^T \hat{A}^{-1} B_r \\ \frac{4\omega}{2+\omega} [\hat{Q}^{-1}]_r B_r^T & I_r - \frac{8\omega^2}{4-\omega^2} [\hat{Q}^{-1}]_r B_r^T \hat{A}^{-1} B_r \end{pmatrix}$$

Then from (22), $\nu(\mathcal{T}) = \nu(\hat{\mathcal{T}}) < 1$ holds if and only if $\rho(\tilde{\mathcal{T}}) < 1$. Note that $\tilde{\mathcal{T}}$ can be viewed as the iteration matrix of the ISSOR iteration method [2] applied to the nonsingular saddle point problem with the following coefficient matrix

$$\begin{pmatrix} \hat{A} & B_r \\ -B_r^T & 0 \end{pmatrix},$$

for the preconditioner $\hat{Q} = V^T Q V$. One can finish the proof off by Theorem 2 in [2]. □

To present the semi-convergence properties of the ISSORS method, by making use of the aforementioned Lemmas 3.2 and 3.3, we summarize and state following Theorem.

Theorem 3.4. *Let $A \in \mathbb{R}^{m\times m}$ be symmetric positive definite, $B \in \mathbb{R}^{m\times n}$ be rank deficient, and ω fulfilled in (16) in Lemma 3.3. Then the ISSORS iteration method (6) is semi-convergent for solving the singular saddle point problem (1).*

4 Conclusion

An extension of the improved symmetric successive over-relaxation method has been given for singular saddle point problems. The method involves one preconditioning matrix for clustering eigenvalues of the iteration matrix of ISSORS method. Furthermore, sufficient conditions has been given for the semi-convergence of the proposed ISSORS method.

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