



# On the Uzawa Accelerated Over-Relaxation Iterative Method for the Elliptic PDE-Constrained Optimization Problems

Mohammad Mahdi Izadkhah<sup>1</sup>

<sup>1</sup>Department of Computer Science, Faculty of Computer and Industrial Engineering, Birjand University of Technology, Birjand, Iran; E-mail: izadkhah@birjandut.ac.ir

## ABSTRACT

In this paper, we give Uzawa variant of accelerated over-relaxation(UAOR) iterative method for solving the elliptic PDE-constrained optimization problems. A functional equation, relates the involved parameters and the eigenvalues of the iteration matrix of the UAOR method. Numerical results are presented to show efficiency and accuracy of the proposed UAOR method.

**Keywords :** Elliptic PDE-Constrained, Optimization Problems, Uzawa Method.

## 1. INTRODUCTION

Consider the following linear elliptic distributed optimal control problems:

$$\min_{u, f} \quad \frac{1}{2} \|u - u_*\|_{L^2(\Omega)}^2 + \beta \|f\|_{L^2(\Omega)}^2, \quad (1)$$

$$s.t. \quad \Delta u = f, \quad \text{in } \Omega, \quad (2)$$

$$u = g \quad \text{on } \partial\Omega_1 \quad \text{and} \quad \frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega_2, \quad (3)$$

where,  $\partial\Omega$  is the boundary of domain  $\Omega$  in  $\mathbb{R}^2$  or,  $\mathbb{R}^3$ ,  $\partial\Omega := \partial\Omega_1 \cup \partial\Omega_2$  and  $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$ . Here  $u$  is the state variable to be computed and  $f$  is the control function, distributed on the whole domain of definition  $\Omega$ . Moreover,  $\beta > 0$  is a regularization parameter and  $u_*$  is a given function representing the desired state. Rees et al. in [3,4] used the discretize-then-optimize approach to transform the problem (1)-(3) to a saddle point system. In fact, by applying the Galerkin finite-element method to the weak formulation of Eqs. (1)-(3), we obtain the following finite-dimensional optimization problem

$$\min_{\mathbf{u}, \mathbf{f}} \quad \frac{1}{2} \mathbf{u}^T M \mathbf{u} - \mathbf{u}^T \mathbf{b} + \|\mathbf{u}_*\|_2^2 + \beta \mathbf{f}^T M \mathbf{f}, \quad (4)$$

$$s.t. \quad K \mathbf{u} = M \mathbf{f} + \mathbf{d},$$

Where  $M \in \mathbb{R}^{m \times m}$  is the mass matrix and  $K \in \mathbb{R}^{m \times m}$  is the stiffness matrix, the entries of the vector  $\mathbf{d} \in \mathbb{R}^m$  come from the boundary values of the discrete solution, and  $\mathbf{b} \in \mathbb{R}^m$ , is the Galerkin projection of the discrete state  $u^*$  [1]. If we put the Lagrange multiplier technique to the minimization problem (4), the following system of linear equations

$$\mathcal{A} \mathbf{x} = \begin{pmatrix} 2\beta M & 0 & M \\ 0 & M & K^T \\ -M & K & 0 \end{pmatrix} \begin{pmatrix} \mathbf{f} \\ \mathbf{u} \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{b} \\ \mathbf{d} \end{pmatrix} = \mathbf{g} \quad (5)$$

is obtained, where  $\lambda \in \mathbb{R}^m$  is a vector of Lagrange multipliers. Both of the matrices  $M$  and  $K$ , and hence also the matrix  $\mathcal{A}$ , are sparse. Moreover, the matrix  $M$  is symmetric positive definite.



In this paper, we reduce the system (5) to a block two-by-two system and then apply Uzawa acceleration overrelaxation iterative method for solving the system.

## 2. REDUCTION TO $2 \times 2$ BLOCK SYSTEM AND SOLUTION

Rewriting the system (5) in terms of  $\mathbf{u}$  and  $\lambda$ , we have a system of linear equations in the generalized saddle point problem of the form

$$\mathcal{B}\hat{\mathbf{x}} = \begin{pmatrix} M & K^T \\ K & -\frac{1}{2\beta}M \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{d} \end{pmatrix}.$$

As proposed in [5], this system can be transform to the following equivalent system

$$\hat{\mathcal{B}}\hat{\mathbf{x}} = \begin{pmatrix} M & -\hat{K}^T \\ \hat{K} & M \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \hat{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \hat{\mathbf{d}} \end{pmatrix} = \hat{\mathbf{g}}. \quad (6)$$

where  $\hat{\lambda} = -\frac{1}{\sqrt{2\beta}}\lambda$ ,  $\hat{K} = \sqrt{2\beta}K$  and  $\hat{\mathbf{d}} = \sqrt{2\beta}\mathbf{d}$ .

Base on the method given in [2], we propose here, Uzawa accelerated over-relaxation (UAOR) iterative method to solve the generalized saddle point system (6). To this end, we consider the following splitting of the coefficient matrix  $\hat{\mathcal{B}} = \mathcal{D} - \mathcal{L} - \mathcal{U}$ , where

$$\mathcal{D} = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} 0 & 0 \\ -\hat{K} & 0 \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} 0 & \hat{K}^T \\ 0 & 0 \end{pmatrix}. \quad (7)$$

We consider two relaxation parameters  $\omega$  and  $r$  as. Now, by using the splitting (7) and the relaxation parameters  $\omega$  and  $r$ , we present UAOR which takes the following form

$$(\mathcal{D} - r\mathcal{L})\hat{\mathbf{x}}^{(k+1)} = ((1 - \omega)\mathcal{D} + (\omega - r)\mathcal{L} + \omega\mathcal{U})\hat{\mathbf{x}}^{(k)} + \omega\hat{\mathbf{g}}.$$

So we have

$$\hat{\mathbf{x}}^{(k+1)} = H(\omega, r)\hat{\mathbf{x}}^{(k)} + F(\omega, r) \quad (8)$$

where

$$H(\omega, r) = \begin{pmatrix} M & 0 \\ r\hat{K} & M \end{pmatrix}^{-1} \begin{pmatrix} (1 - \omega)M & \omega\hat{K}^T \\ -(\omega - r)\hat{K} & (1 - \omega)M \end{pmatrix} \quad (9)$$

and

$$F(\omega, r) = \omega \begin{pmatrix} M & 0 \\ r\hat{K} & M \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{b} \\ \hat{\mathbf{d}} \end{pmatrix} \quad (10)$$

**Remark.** The UAOR method (8) with  $r = \omega$  reduces to the Uzawa variant of successive over-relaxation (USOR) method.

We assume that the matrices  $M$  and  $\hat{K}$  are symmetric positive definite (SPD) and symmetric positive semi definite (SPSD), respectively.

We describe the algorithm of UAOR method (8) as follows.

### Algorithm UAOR

Given initial vectors  $\mathbf{u}^{(0)} \in \mathbb{R}^m$ ,  $\hat{\lambda}^{(0)} \in \mathbb{R}^m$  and relaxation parameters  $\omega$  and  $r$ . For  $k = 0, 1, 2, \dots$  until iteration sequence  $\{[(\mathbf{u}^{(k)})^T, (\hat{\lambda}^{(k)})^T]^T\}_{k=0}^{\infty}$  is convergent, compute

$$\begin{aligned}\mathbf{u}^{(k+1)} &= (1 - \omega)\mathbf{u}^{(k)} + M^{-1}(\omega\hat{K}\hat{\lambda}^{(k)} + \omega\mathbf{b}), \\ \hat{\lambda}^{(k+1)} &= (1 - \omega)\hat{\lambda}^{(k)} + M^{-1}((r - \omega)\hat{K}\mathbf{u}^{(k)} - r\hat{K}\mathbf{u}^{(k+1)} + \omega\hat{\mathbf{d}}).\end{aligned}$$

It could be naturally induced a splitting preconditioner  $P_{\omega,r}$  for Krylov subspace methods like GMRES, corresponds to the following AOR-splitting

$$\begin{pmatrix} M & -\hat{K}^T \\ \hat{K} & M \end{pmatrix} = P_{\omega,r} - Q_{\omega,r}$$

where

$$P_{\omega,r} = \frac{1}{\omega} \begin{pmatrix} M & 0 \\ r\hat{K} & M \end{pmatrix}, \quad Q_{\omega,r} = \frac{1}{\omega} \begin{pmatrix} (1 - \omega)M & \omega\hat{K}^T \\ -(\omega - r)\hat{K} & (1 - \omega)M \end{pmatrix}.$$

We call this preconditioner the AOR-splitting preconditioner for the problem (3). At each step of the UAOR iteration in the aforementioned Algorithm or applying the AOR-splitting preconditioner  $P_{\omega,r}$  within a Krylov subspace method, we need to solve a linear system with  $M$  as the coefficient matrix. To solve symmetric positive definite system of linear equations with  $M$  as the coefficient matrix, one can use the sparse Cholesky factorization incorporated with the symmetric approximate minimum degree reordering. For this end, we have used symamd command of Matlab 8.5.

### 3. FUNCTIONAL EQUATION

Let  $\mu$  be an eigenvalue of  $H(\omega, r)$ , and  $[x^T, y^T]^T \neq 0$  be the corresponding eigenvector. Then

$$\begin{aligned}(1 - \omega - \mu)Mx &= -\omega\hat{K}y \\ (r - r\mu - \omega)\hat{K}x &= (\mu + \omega - 1)My.\end{aligned}\tag{11}$$

**Theorem 1.** For any eigenvalue  $\mu \neq (1 - \omega)$ , of  $H(\omega, r)$ , there is an eigenvalue  $\xi$  of  $(M^{-1}\hat{K})^2$ , so that  $\mu, \xi$  and  $\omega$  satisfy the following functional equation

$$\omega(r - r\mu - \omega)\xi = (\mu + \omega - 1)^2\tag{12}$$

Conversely, for any eigenvalue  $\xi$  of  $(M^{-1}\hat{K})^2$ , if  $\mu$  and  $\omega$  satisfy the above functional equation, then  $\mu$  is an eigenvalue of  $H(\omega, r)$ .

**Corollary 1.** If we let  $\rho(H(\omega, r))$  be the spectral radius of the UAOR iteration matrix, then we have

$$\rho(H(\omega, r)) \geq |1 - \omega|$$

By the above corollary, the necessary condition for the convergence of the UAOR method is  $0 < \omega < 2$ .

#### 4. NUMERICAL RESULTS

In this section, we describe some numerical experiments which were carried out in order to show efficiency of the UAOR method for different values of parameters  $\omega$  and  $r$ . Here we consider the linear system of equations (6) according to the following data.

$$h = \frac{1}{m+1}, \quad Z = \frac{1}{h^2} \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m \times m}, \quad V = \frac{1}{h^2} \text{tridiag}(-1, 1, 0) \in \mathbb{R}^{m \times m}$$

$$M = \begin{pmatrix} I \otimes Z + Z \otimes I & 0 \\ 0 & I \otimes Z + Z \otimes I \end{pmatrix}, \quad \hat{K} = \sqrt{2\beta} \begin{pmatrix} I \otimes V + V \otimes I & 0 \\ 0 & I \otimes V + V \otimes I \end{pmatrix}$$

We consider  $\hat{\mathbf{b}}$  and  $\hat{\mathbf{d}}$  in (6) so the exact solution of the augmented system of linear equations (6) is

$$\begin{pmatrix} \mathbf{u}^* \\ \hat{\lambda}^* \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^{4m^2}$$

For stopping criterion and in order to study our results, we will use the relative error norm of the approximate solution.

$$ERR = \frac{\sqrt{\|\mathbf{u}^{(k)} - \mathbf{u}^*\|_2^2 + \|\hat{\lambda}^{(k)} - \hat{\lambda}^*\|_2^2}}{\sqrt{\|\mathbf{u}^*\|_2^2 + \|\hat{\lambda}^*\|_2^2}} < 10^{-6}$$

The numerical results for the UAOR method and different values  $m$  of are presented in Table 1. Also depiction of the relative error norms for different values of the parameters  $\omega$  and  $r$  and  $\beta = 10^{-2}$  are showed in Figure 1. CPU-Time and IT(iteration number) are presented in Table 1. Form the information of numerical results in Table 1 and Figure 1, it can be noted that the presented UAOR method is an efficient iterative method for solving elliptic PDE-constrained optimization problem (1).

**Table 1- Numerical Results of UAOR for  $\omega = 1.25$  and  $r = 0.75$**

UAOR method for different $m$	IT	CPU-Time(second)	ERR
$m = 8$	14	0.004	6.6706e-07
$m = 16$	19	0.1312	6.9980e-07
$m = 32$	38	7.5346	8.8518e-07

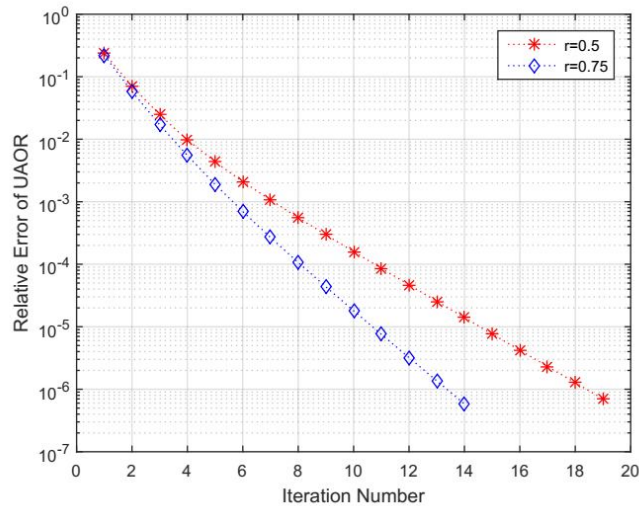


Figure 1. Numerical Results of UAOR for  $m = 16$  and  $\omega = 1.25$  for different values of  $r$

## 5. CONCLUSIONS

In the current paper, Uzawa accelerated over-relaxation(UAOR) iterative method has been presented for solving the elliptic PDE-constrained optimization problems. Numerical results have confirmed the efficiency and accuracy of the proposed UAOR method.

## 6. REFERENCES

- [1] Ke, Y.-F., Ma, C.-F., "Some preconditioners for elliptic PDE-constrained optimization problems", *Comput. Math. Appl.*, No. 75, pp. 2795 – 2813, 2018.
- [2] Martins, M.M., Yousif, W., Santos, J.L "A variant of the AOR method for augmented systems", *SIAM J. Math. Comput.* No. 277, pp. 399 – 417, 2012.
- [3] Rees, T., Dollar, H.S., Wathen, A.J. "Optimal solvers for PDE-constrained optimization", *SIAM J. Sci. Comput.* , No. 32, pp. 271 – 298, 2010.
- [4] Rees, T., Stoll, M., "Block-triangular preconditioners for PDE-constrained optimization", *Numer. Linear Algebra Appl.*, No. 17, pp. 977 – 996, 2010.
- [5] Salkuyeh, D.K., "On the preconditioning of elliptic PDE-constrained optimization problems", *The 50 th Annual Mathematics Conference*, Shiraz University, 26-29 August, 2019.