A Hybrid of Diagonal Preconditioner and Shift-Splitting Method for Double Saddle Point Problems

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Abstract. In this paper, we study a hybrid of diagonal preconditioner and shift-splitting method for numerical solution of double saddle point problems. Theoretical analysis shows that the proposed iterative method is unconditionally convergent. Some numerical results are presented to clarify the effectiveness and accuracy of the presented preconditioner for Krylov subspace method, like GMRES.
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1. Introduction

In recent years, there has been a growing interest to the saddle point problems in the field of numerical linear algebra. This kind of linear systems arise in a great deal of sciences, such as nonlinear constrained optimization, finite element approximation for solving the Navier-Stockes equation, incompressible elasticity, constrained least squares problems, and so forth [2, 3].

Double saddle point problems has been considered as the following large and sparse form

$$\mathcal{A}\mathbf{u} \equiv \begin{pmatrix} A & B & C \\ -B^T & 0 & 0 \\ -C^T & 0 & D \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \equiv \mathbf{b}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{p\times p}$ are symmetric positive definite(SPD), $B \in \mathbb{R}^{n\times m}$ with $\operatorname{rank}(B) = m < n$, $C \in \mathbb{R}^{n\times p}$, $x, b_1 \in \mathbb{R}^n$, $y, b_2 \in \mathbb{R}^m$ and $z, b_3 \in \mathbb{R}^p$. The following proposition given in [2] provides a necessary and sufficient condition for the invertibility of the matrix $\mathcal{A}$ in the case that the $(1,1)$-block and $(3,3)$-block are both SPD.

Proposition 1.1. *Assume that A and D are symmetric positive definite (SPD). Then matrix $\mathcal{A}$ in (1) is invertible if and only if B has full column rank.*

Both Uzawa-type stationary methods and block preconditioned Krylov subspace methods are discussed in [2] for double saddle point problem (1).

The reminder of this paper is organized as follows. In Section 2, we preset a hybrid of diagonal preconditioner and shift-splitting iteration method. In Section

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3, theoretical investigation will provide for the given method. Some numerical results are given in Section 4, to clarify the effectiveness and accuracy of the presented preconditioner for Krylov subspace methods.

2. Hybrid of Diagonal Preconditioner and Shift-Splitting

To establish the properties of preconditioned shift-splitting iterative method, we propose a diagonal preconditioner to system (1) as

$$P = \begin{pmatrix} A & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & D \end{pmatrix},$$

where $Q \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix. So, we define

$$\bar{\mathcal{A}} = P^{-\frac{1}{2}}\mathcal{A}P^{-\frac{1}{2}} = \begin{pmatrix} I & \bar{B} & \bar{C} \\ -\bar{B}^T & I & 0 \\ -\bar{C}^T & 0 & I \end{pmatrix}, \quad \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = P^{\frac{1}{2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} A^{\frac{1}{2}}x \\ Q^{\frac{1}{2}}y \\ D^{\frac{1}{2}}z \end{pmatrix},$$

and

$$\begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{pmatrix} = P^{-\frac{1}{2}}\mathbf{b} = \begin{pmatrix} A^{-\frac{1}{2}}b_1 \\ Q^{-\frac{1}{2}}b_2 \\ D^{-\frac{1}{2}}b_3 \end{pmatrix},$$

where $\bar{B} = A^{-\frac{1}{2}}BQ^{-\frac{1}{2}} \in \mathbb{R}^{n\times m}$ also has full column rank, and $\bar{C} = A^{-\frac{1}{2}}CD^{-\frac{1}{2}} \in \mathbb{R}^{n\times p}$. Then, system (1) can be transformed into a new equivalent one

$$\begin{pmatrix} I & \bar{B} & \bar{C} \\ -\bar{B}^T & 0 & 0 \\ -\bar{C}^T & 0 & I \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{pmatrix}. \tag{2}$$

Applying shift-splitting method given in [1, 4] into (2), we obtain

$$\frac{1}{2}\begin{pmatrix} (1+\alpha)I & \bar{B} & \bar{C} \\ -\bar{B}^T & \alpha I & 0 \\ -\bar{C}^T & 0 & (1+\alpha)I \end{pmatrix} \begin{pmatrix} \bar{x}^{k+1} \\ \bar{y}^{k+1} \\ \bar{z}^{k+1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (\alpha-1)I & -\bar{B} & -\bar{C} \\ \bar{B}^T & \alpha I & 0 \\ \bar{C}^T & 0 & (\alpha-1)I \end{pmatrix} \begin{pmatrix} \bar{x}^{k} \\ \bar{y}^{k} \\ \bar{z}^{k} \end{pmatrix} + \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{pmatrix}.$$

It then follows immediately that in the original variables,

$$\frac{1}{2}\begin{pmatrix} (1+\alpha)A & B & C \\ -B^T & \alpha Q & 0 \\ -C^T & 0 & (1+\alpha)D \end{pmatrix} \begin{pmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (\alpha-1)A & -B & -C \\ B^T & \alpha Q & 0 \\ C^T & 0 & (\alpha-1)D \end{pmatrix} \begin{pmatrix} x^{k} \\ y^{k} \\ z^{k} \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

It could be naturally induced a splitting preconditioner $\mathcal{P}_{DPSS}$ for Krylov subspace methods, corresponds to the diagonal preconditioned shift-splitting (DPSS) iteration as

$$\mathcal{P}_{DPSS} = \frac{1}{2}\begin{pmatrix} (1+\alpha)A & B & C \\ -B^T & \alpha Q & 0 \\ -C^T & 0 & (1+\alpha)D \end{pmatrix}.$$

We can do the following matrix factorization for splitting preconditioner $\mathcal{P}_{DPSS}$.

$$\mathcal{P}_{DPSS} = \frac{1}{2}\begin{pmatrix} I & \frac{1}{\alpha}BQ^{-1} & \frac{1}{1+\alpha}CD^{-1} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} S & 0 & 0 \\ 0 & \alpha Q & 0 \\ 0 & 0 & (1+\alpha)D \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ -\frac{1}{\alpha}Q^{-1}B^T & I & 0 \\ -\frac{1}{1+\alpha}D^{-1}C^T & 0 & I \end{pmatrix},$$

where $S = (1+\alpha)A + \frac{1}{\alpha}BQ^{-1}B^T + \frac{1}{1+\alpha}CD^{-1}C^T \in \mathbb{R}^{n\times n}$. At each step of the DPSS iteration or applying the DPSS preconditioner $\mathcal{P}_{DPSS}$ within a Krylov subspace method, we need to solve a linear system as $\mathcal{P}_{DPSS}\mathbf{z}^{(k)} = \mathbf{r}^{(k)}$ for a given residual vector $\mathbf{r}^{(k)}$ at each step. In the following Algorithm, Let us consider $\mathbf{r}^{(k)} = [r_1^T, r_2^T, r_3^T]^T$ and $\mathbf{z}^{(k)} = [z_1^T, z_2^T, z_3^T]^T$, where $r_1, z_1 \in \mathbb{R}^n$, $r_2, z_2 \in \mathbb{R}^m$ and $r_3, z_3 \in \mathbb{R}^p$.

Algorithm 1. Diagonal Preconditioned Shift-Splitting Iteration Method

(1) Set $k := 0$. Given inital guess $\mathbf{u}^{(0)}$ and $\alpha > 0$. Choose $\epsilon > 0$ as the precision, and k_{max} as the maximum iteration. Set $\mathbf{r}^{(0)} = \mathbf{b} - \mathcal{A}\mathbf{u}^{(0)}$, and its block entries r_1, r_2 and r_3 as defined in advance.
(2) For $\mathbf{u}^{(k)} \in \mathbb{R}^{m+n+p}$, and associated residual $\mathbf{r}^{(k)}$, if $\frac{\|\mathbf{r}^{(k)}\|_2}{\|\mathbf{r}^{(0)}\|_2} \geq \epsilon$ or $k \leq k_{max}$ continue, goto Step 3, else STOP.
(3) Solve $Dw = \frac{2}{1+\alpha} r_3$.
(4) Solve $Qy = r_2$.
(5) $w_1 = 2\left(r_1 - \frac{1}{\alpha}By\right) - Cw$.
(6) Solve $Sz_1 = w_1$.
(7) Solve $Qz_2 = \frac{1}{\alpha}(B^T z_1 + 2r_2)$.
(8) Solve $Dv = \frac{1}{1+\alpha} C^T z_1$.
(9) $z_3 = v + w$.
(10) Let $\mathbf{z}^{(k)} = [z_1^T, z_2^T, z_3^T]^T$, compute $\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + \mathbf{z}^{(k)}$ and set $k := k+1$, goto Step 2.

Remark 2.1. From Algorithm 1, we can see that at each iteration, it is required to solve linear systems with the coefficient matrices Q, D and $(1+\alpha)A + \frac{1}{\alpha}BQ^{-1}B^T + \frac{1}{1+\alpha}CD^{-1}C^T$. Fortunately, the aforementioned matrices are symmetric positive definite, for all $\alpha > 0$.

Remark 2.2. The symmetric positive definite matrix $Q \in \mathbb{R}^{m\times m}$ should be chosen such that the linear system with coefficient matrix Q is easily solvable, and the singular values of the matrix $A^{-\frac{1}{2}}BQ^{-\frac{1}{2}} \in \mathbb{R}^{n\times m}$ are tightly clustered, or in other words, Q should be a good preconditioner to the matrix $B^T A^{-1} B \in \mathbb{R}^{m\times m}$.

3. Convergence Analysis of DPSS Method

Now, we turn to study the convergence of the diagonal preconditioned shift-splitting iteration method. Note that the iteration matrix of the proposed method is

$$(3)\quad \mathcal{T}_{DPSS} = \begin{pmatrix} (1+\alpha)A & B & C \\ -B^T & \alpha Q & 0 \\ -C^T & 0 & (1+\alpha)D \end{pmatrix}^{-1} \begin{pmatrix} (\alpha-1)A & -B & -C \\ B^T & \alpha Q & 0 \\ C^T & 0 & (\alpha-1)D \end{pmatrix}.$$

Let $\rho(\mathcal{T}_{DPSS})$ denote the spectral radius of $\mathcal{T}_{DPSS}$. To study the convergence of the diagonal preconditioned shift-splitting iteration, we first give the following lemma.

Lemma 3.1. *Let A and D be symmetric positive definite matrices, and B have full row rank. Let $\mathcal{T}_{DPSS}$ be defined as in (3). If λ is an eigenvalue of $\mathcal{T}_{DPSS}$, then $\lambda \neq \pm 1$.*

Next theorem could be proved similar in [4] for the convergence of the diagonal preconditioned shift splitting scheme proposed in this paper.

Theorem 3.2. *Let $A \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{p\times p}$ be symmetric positive definite matrices and $B \in \mathbb{R}^{n\times m}$ has full row rank, and let α be a positive number. Then, we have*

$$\rho(\mathcal{T}_{DPSS}) < 1, \quad \text{for all } \alpha > 0.$$

We propose using the Krylov subspace method like GMRES, or its restarted version GMRES(m) to accelerate the convergence of the iteration. It is easy to see that the linear system $\mathcal{A}u = b$ is equivalent to the linear system [5]

$$(I - \mathcal{T}_{DPSS})\mathbf{u} = \mathcal{P}_{DPSS}^{-1}\mathcal{A}\mathbf{u} = \mathcal{P}_{DPSS}^{-1}\mathbf{b}.$$

4. Numerical Results

In practical computations, we use left preconditioning with restarted GMRES($\sharp$) as the Krylov subspace method. Here, the integer $\sharp$ in GMRES($\sharp$) denotes that the algorithm is restarted after every $\sharp$ iterations. In this paper, we take $\sharp = 30$. All runs are started from the initial zero vector and terminated if the current iterations satisfy $ERR = \frac{\|\mathbf{r}^{(k)}\|_2}{\|\mathbf{r}^{(0)}\|_2} \le 10^{-6}$, or if the prescribed iteration number $k_{\max} = 5000$ is exceeded. All runs are performed in MATLAB R2015a on an Intel Core i3 Laptop with 4G RAM. We consider two cases of the DPSS method as follows:
- Case I: $Q = I_m$,
- Case II: $Q = \beta B^T B$ for $\beta = 0.001$.

Example 4.1. By the finite difference scheme of the Stokes problem, the submatrices of the coefficient matrix in the double saddle point problems have the following form

$$A = \begin{pmatrix} I\otimes\ T + T\otimes I & 0 \\ 0 & I\otimes\ T + T\otimes I \end{pmatrix} \in \mathbb{R}^{2q^2\times 2q^2}, \quad B = \begin{pmatrix} I\otimes\ F \\ F\otimes\ I \end{pmatrix} \in \mathbb{R}^{2q^2\times q^2},$$

$$D = I\otimes\ T + T\otimes I \in \mathbb{R}^{q^2\times q^2}, \qquad C = \begin{pmatrix} I\otimes\ F \\ F\otimes\ I \end{pmatrix} \in \mathbb{R}^{2q^2\times q^2},$$

where $T = \frac{\nu}{h^2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{q\times q}$, $F = \frac{1}{h}\text{tridiag}(-1, 1, 0) \in \mathbb{R}^{q\times q}$, with $\otimes$ being the Kronecker product symbol and $h = \frac{1}{q+1}$ the discretization mesh size.

In Tables 1 and 2, we list the numerical results corresponding to the two ν, i.e. $\nu = 0.1, 0.01$. For each ν, three different q are used, i.e. $q = 8, 16, 24$. The parameter α in the $\mathcal{P}_{DPSS}$ is taken the same the viscosity ν. In these tables, $\mathcal{I}$, $\mathcal{P}_{DPSS}$ and $\mathcal{P}_{HSS}$ denote the GMRES(30) method without preconditioning, with the left DPSS preconditioning and with the left HSS preconditioning, respectively. IT, CPU and ERR stand for the iteration numbers, the elapsed CPU times (in seconds) and the relative error, respectively. To demonstrate efficiency of diagonal preconditioned shift-splitting method, the HSS preconditioner is considered as follows

$$\mathcal{P}_{HSS} = \begin{pmatrix} \alpha I + A & 0 & 0 \\ 0 & \alpha I & 0 \\ 0 & 0 & \alpha I + D \end{pmatrix} \begin{pmatrix} \alpha I & B & C \\ -B^T & \alpha I & 0 \\ -C^T & 0 & \alpha I \end{pmatrix}.$$

TABLE 1. Numerical results for solving Example 4.1 with $\nu = 0.1$.

Grid		$\mathcal{I}$	$\mathcal{P}_{DPSS}$		$\mathcal{P}_{HSS}$
			Case I	Case II	
8×8					
	IT	7(6)	1(4)	1(3)	3(29)
	CPU	0.347	0.031	0.023	2.047
	ERR	9.7061e-07	5.4733e-07	5.7328e-07	6.3789e-07
16×16					
	IT	12(21)	1(5)	1(4)	6(25)
	CPU	7.245	1.500	1.400	111.821
	ERR	9.7001e-07	3.6912e-08	4.0800e-08	9.97071e-07
24×24					
	IT	24(27)	1(5)	1(4)	†
	CPU	66.679	12.513	9.156	†
	ERR	9.9045e-07	6.3548e-08	1.7903e-07	†

TABLE 2. Numerical results for solving Example 4.1 with $\nu = 0.01$.

Grid		$\mathcal{I}$	$\mathcal{P}_{DPSS}$		$\mathcal{P}_{HSS}$
			Case I	Case II	
8×8					
	IT	47(26)	1(2)	1(2)	3(29)
	CPU	2.757	0.013	0.014	2.103
	ERR	9.9691e-07	4.3954e-07	3.7612e-09	8.8729e-07
16×16					
	IT	95(21)	1(2)	1(2)	7(3)
	CPU	60.123	0.477	0.461	129.335
	ERR	9.9886e-07	5.3702e-07	7.4961e-09	8.60139e-07
24×24					
	IT	124(18)	1(2)	1(2)	†
	CPU	352.326	4.067	4.205	†
	ERR	9.9951e-07	6.3462e-07	1.2512e-08	†

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