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Convergence region of the generalized accelerated overrelaxation method for double saddle point problems

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Abstract

This paper is devoted to the generalize accelerated overrelaxation iterative method for solving a class of double saddle point problems. Also, we study convergence region of the proposed method and then some numerical results are given to demonstrate the efficiency of the presented method.

Keywords: AOR iterative method, Saddle point problem, convergence analysis

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1 Introduction

We consider a class of double saddle point problems as the following large and sparse form

$$\mathcal{A}\mathbf{u} \equiv \begin{pmatrix} A & B & C \\ -B^T & 0 & 0 \\ -C^T & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \equiv \mathbf{b}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite(SPD) matrix, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{n\times p}$ have full column ranks, $x, b_1 \in \mathbb{R}^n$, $y, b_2 \in \mathbb{R}^m$ and $z, b_3 \in \mathbb{R}^p$. For real eigenvalues of A, we use $\lambda_{min}(A)$ and $\lambda_{max}(A)$ to denote the minimum and maximum eigenvalues of A, respectively. Moreover, the notations $\mathrm{Ran}(A)$ and $\rho(A)$ stand for the range and the spectral radius of A, respectively. For given vectors x, y and z of dimension n, m and p, respectively, $\mathbf{u} = (x; y; z)$ will denote a column vector of dimension $n+m+p$. Linear systems of the form (1) arise from mixed finite element approximation of the potential fluid flow problems ; see [1, 2] and the references therein for detailed descriptions of these problems.

The following Proposition given in [2] represents the necessary and sufficient condition of the invertibility of the coefficient matrix $\mathcal{A}$ in (1).

Proposition 1.1. *Let A be a SPD matrix and assume that B and C have full column ranks. Then a necessary and sufficient condition for the invertibility of the matrix $\mathcal{A}$ in (1) is that* $\mathrm{Ran}(B) \cap \mathrm{Ran}(C) = \{0\}$.

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2 Convergence region of DGAOR method

We propose generalized accelerated overrelaxation(AOR) iterative method for solving (1), based on the following splitting

$$\mathcal{A} = \mathcal{D} - \mathcal{E} - \mathcal{F},$$

where

$$\mathcal{D} = \begin{pmatrix} A & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & D \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} 0 & 0 & 0 \\ B^T & 0 & 0 \\ C^T & 0 & 0 \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} 0 & -B & -C \\ 0 & Q & 0 \\ 0 & 0 & D \end{pmatrix}.$$

Here Q and D are SPD matrices. The iteration matrix of the generalized AOR method is defined as

$$\mathcal{M}_{r,\omega} = \begin{pmatrix} (1-\omega)I & -\omega A^{-1}B & -\omega A^{-1}B \\ \omega(1-r)Q^{-1}B^T & I - r\omega Q^{-1}B^T A^{-1}B & 0 \\ \omega(1-r)D^{-1}C^T & 0 & I - r\omega D^{-1}C^T A^{-1}C \end{pmatrix}. \tag{2}$$

Note that if $\omega = 0$, then the proposed generalized AOR method diverges no matter what value the accelerated parameter r takes. In view of this, we will assume $\omega \neq 0$ and so the generalized AOR method to solve double saddle point system (1) (DGAOR) can be defined by the following form

$$\mathbf{u}^{(k+1)} = \mathcal{M}_{r,\omega}\mathbf{u}^{(k)} + \mathbf{c}, \quad k = 0, 1, 2, \ldots, \tag{3}$$

where

$$\mathbf{c} = \begin{pmatrix} A^{-1}b_1 \\ Q^{-1}(rB^T A^{-1}b_1 + b_2) \\ D^{-1}(rC^T A^{-1}b_1 + b_3) \end{pmatrix},$$

and $\mathbf{u}^{(0)} \in \mathbb{R}^{m+n+p}$ is the initial guess.

In the following Lemma, we discuss about $\lambda = 1$ as an eigenvalue of the iteration matrix $\mathcal{M}_{r,\omega}$ of DGAOR method to analyse the spectral properties of the iteration matrix. In fact, we show that $\lambda = 1$ could not be an eigenvalue of $\mathcal{M}_{r,\omega}$.

Lemma 2.1. *If λ is an eigenvalue of the iteration matrix $\mathcal{M}_{r,\omega}$ of DGAOR method corresponding to the eigenvector $\mathbf{u} = (x; y; z)$, then x and z are not equal to zero, simultaneously, and $\lambda \neq 1$.*

Proof. If we let λ be an eigenvalue of $\mathcal{M}_{r,\omega}$ and $\mathbf{u} = (x; y; z)$ be the corresponding eigenvector, then we have

$$(1 - \omega - \lambda)x = \omega A^{-1}(By + Cz), \tag{4}$$

$$(\omega - r + r\lambda)Q^{-1}B^T x = (\lambda - 1)y, \tag{5}$$

$$(\omega - r + r\lambda)D^{-1}C^T x = (\lambda - 1)z. \tag{6}$$

If we set $x = 0$ and $z = 0$, then (4) implies that $By = 0$. Since B has full column rank, so $y = 0$ which is a contradiction. Let $\lambda = 1$, and the associated eigenvector $\mathbf{u} = (x; y; z)$. Then, by equations (4)-(6) we have

$$A^{-1}(By + Cz) = -x, \quad Q^{-1}B^T x = 0, \quad D^{-1}C^T x = z.$$

This is the problem $\mathcal{A}\mathbf{u} = 0$, and by Proposition5 1.1 we have $\mathbf{u} = 0$, which is a contradiction. $\square$

In the following Lemma, we investigate the multiplicity of the eigenvalues of $\mathcal{M}_{r,\omega}$.

Lemma 2.2. *If $r = 1$, then $\lambda = 1 - \omega$ is an eigenvalue of $\mathcal{M}_{r,\omega}$ with multiplicity of at least m. If $r \neq 1$, then $\lambda = 1 - \omega$ is an eigenvalue of $\mathcal{M}_{r,\omega}$ if and only if $n > m$; in this case the multiplicity of λ is $n - m - p$.*

Proof. By definition (2) of $\mathcal{M}_{r,\omega}$, it can be deduced that for $r = 1$, $\lambda = 1 - \omega$ is an eigenvalue of $\mathcal{M}_{r,\omega}$ with multiplicity of at least m. Now, we assume $r \neq 1$. By equations (4)-(6) we have

$$(r-1)x^T BQ^{-1}B^T x = 0,$$

which implies $x = 0$ for $n = m$, and $x \neq 0$ for $n > m$. Thus $\lambda = 1 - \omega$ is an eigenvalue of $\mathcal{M}_{r,\omega}$ with multiplicity of $n - (m + p)$. The latter status comes from the fact that the algebraic multiplicity of an eigenvalue is at least equal to the geometrical multiplicity. □

Corollary 2.3. *Let λ be an eigenvalue of the iteration matrix $\mathcal{M}_{r,\omega}$ and the associated eigenvector is $\mathbf{u} = (x; y; z)$. If $\lambda \neq 1 - \omega$, then $y \neq 0$ or $z \neq 0$.*

In the following Theorem, we give a functional equation between parameters ω and r, and λ as an eigenvalue of $\mathcal{M}_{r,\omega}$ for the next investigation of the convergence region of the parameters in the DGAOR method.

Theorem 2.4. *Let A be a SPD matrix and assume that B and C have full column ranks, such that $\operatorname{Ran}(B) \cap \operatorname{Ran}(C) = \{0\}$. If Q and D are SPD matrices and λ be an eigenvalue of $\mathcal{M}_{r,\omega}$, then for the parameters ω and r, we have*

$$\lambda^2 + ((\omega - 2) + \frac{\gamma + \beta}{\alpha} r\omega)\lambda + (1 - \omega) + \frac{\gamma + \beta}{\alpha}\omega(\omega - r) = 0,$$

where $\alpha = x^* A x$, $\beta = x^* BQ^{-1}B^T x$, and $\gamma = x^* CD^{-1}C^T x$.

Proof. Equations (4)-(6) give

$$\left(A + \frac{\omega(\omega - r + r\lambda)}{(1-\omega-\lambda)(1-\lambda)} CD^{-1}C^T\right) x = \frac{\omega}{1-\omega-\lambda} By, \quad (\omega - r + r\lambda)Q^{-1}B^T x = (\lambda - 1)y,$$

and therefore by definitions of α, β and γ, we have

$$(1-\lambda)(1-\omega-\lambda)\alpha + \omega(\omega - r + r\lambda)(\gamma + \beta) = 0.$$

□

In the following Lemma, we recall a necessary and sufficient condition of both roots of the real quadratic equation to be less than one in modulus.

Lemma 2.5 ([4, Lemma 2.3]). *Both roots of the real quadratic equation $\lambda^2 - b\lambda + c = 0$, are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.*

It follows from Lemma 2.5 that $\rho(\mathcal{M}_{r,\omega}) < 1$ if and only if

$$|1 - \omega + \frac{\gamma + \beta}{\alpha}\omega(\omega - r)| < 1, \tag{7}$$

$$|\omega - 2 + \frac{\gamma + \beta}{\alpha} r\omega| < 1 + 1 - \omega + \frac{\gamma + \beta}{\alpha}\omega(\omega - r). \tag{8}$$

Equations (7) and (8) hold true if we have

$$0 < \omega < 2, \quad \omega - \frac{\alpha}{\gamma + \beta} < r < \frac{\omega}{2}.$$

Consequently, we have the following results.

Theorem 2.6. *Let A be a SPD matrix and assume that B and C have full column ranks, such that $\operatorname{Ran}(B) \cap \operatorname{Ran}(C) = \{0\}$. If $0 < \omega < \min(2, \frac{2\lambda_{min}(A)}{\lambda_{max}(BQ^{-1}B^T) + \lambda_{max}(CD^{-1}C^T)})$ and*

$$\omega - \frac{\lambda_{min}(A)}{\lambda_{max}(BQ^{-1}B^T) + \lambda_{max}(CD^{-1}C^T)} < r < \frac{\omega}{2}$$

then the DGAOR iterative scheme (3) converges to the exact solution of (1).

3 Numerical results

We now describe some numerical experiments were carried out in order to analyze the behaviour of the DGAOR method for different values of the parameter ω and r. The computational study was done in the next problems.

Example 3.1. Let us consider the double saddle point system (1), where the entries of the matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{n\times p}$ are defined as follows

$$A = (a_{ij}) = \begin{cases} i+1, & i=j \\ 1, & |i-j|=1 \\ 0, & otherwise. \end{cases}, \quad B = (b_{ij}) = \begin{cases} j, & i = n-m+j \\ 0, & otherwise. \end{cases}$$

and

$$C = (c_{ij}) = \begin{cases} j, & i=j \\ 0, & otherwise. \end{cases}$$

For this problem, the condition of Proposition 1.1 are satisfied(especially $\mathrm{Ran}(B)\cap\mathrm{Ran}(C) = \{0\}$), hence $\mathcal{A}$ is nonsingular and the double saddle point problem (1) has a unique solution. The vector $\mathbf{b}$ is chosen so that the components of the exact solution $\mathbf{u}$ of (1) have values equal to 1. We choose the preconditioning matrices $Q = B^TB$ and $D = C^TC$ for the DGAOR method. All runs are started with the initial zero vector and terminated if the current iterations satisfy $ERR = \frac{\|\mathbf{r}^{(k)}\|_2}{\|\mathbf{r}^{(0)}\|_2} \leq 10^{-4}$, or if the prescribed iteration number $k_{\max} = 2000$ is exceeded. Here, we define $\mathbf{r}^{(k)}$ as

$$\mathbf{r}^{(k)} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} A & B & C \\ -B^T & 0 & 0 \\ -C^T & 0 & 0 \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix}.$$

Figure 1 shows ERR in terms of ω and r for $n = 50, m = 30, p = 10$(left) and $n = 200, m = 100, p = 60$(right). In view of Figure 1, we can conclude the fact that the minimum ERR is obtained when the parameters ω and r are near the boundary of the convergence region. From the results reported in Table 1, we can conclude that ERR and computational time are important items to demonstrate the efficiency of the DGAOR method in comparison with the SOR-like method($\omega = r$) [3].

Table 1: CPU time, iteration number and values of the parameters n, m, p, ω and r for DGAOR method

					DGAOR method			SOR-like method		
n	m	p	ω	r	minIT	CPU(s)	ERR	minIT	CPU(s)	ERR
50	30	10	1.2538	0.5769	283	0.0313	9.902e-05	285	0.0625	9.996e-05
80	40	20	1.2538	0.5769	427	0.1719	9.980e-05	429	0.2188	9.978e-05
100	50	40	1.2538	0.5769	528	0.2813	9.918e-05	530	0.3281	9.914e-05
300	150	80	1.2538	0.5769	1571	4.2500	9.983e-05	1573	4.4688	9.983e-05

4 Conclusion

In this paper, a generalization of accelerated overrelaxation (AOR) iterative method for solving a class of double saddle point problems have been proposed, where is denoted by DGAOR. The convergence region of the DGAOR method has been analyzed and numerical experiments were given to demonstrate the efficiency of the DGAOR method in comparison with the SOR-like method.

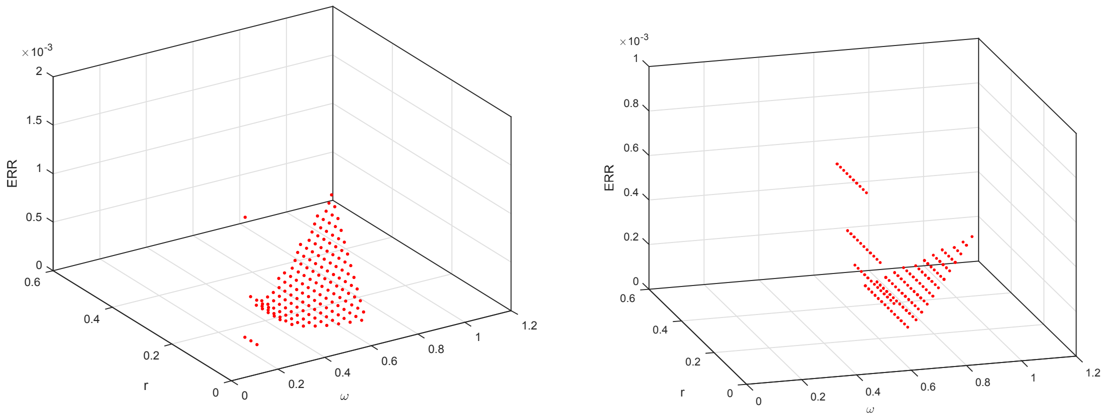

Figure 1: Convergence region for $n = 50, m = 30, p = 10$(left) and $n = 200, m = 100, p = 60$(right)

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