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GEGENBAUER SPECTRAL TREATMENT OF THE TIME FRACTIONAL BLACK-SCHOLES-SCHRODINGER EQUATIONS

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Abstract. In this paper, we study the time fractional Black-Scholes-Schrodinger(TFBSS) equation as a quantum financial model which is used for analyzing fair prices of options in real financial market. We present Gegenbauer spectral method for spatial variable and the time fractional derivative of mentioned equation is approximated by a scheme of order $2-\alpha$ for fractional derivation of order α. The presented method reduces TFBSS to a matrix equation, which can be solved by the restarted global GMRES method.

Keywords: Time fractional Black–Scholes-Schrodinger model, Gegenbauer spectral method, Matrix equation, global GMRES method.
Classification: 34K37, 97N50, 65M70.

1. Introduction

One of the most important and popular financial derivatives in financial marketing is the considerable option. There are various types of mathematical model for option pricing. The balck-Scholes equation, introduced in [6], provided an approximate description of underlying asset price behavior. This equation becomes popular in various kinds of studies such as economics, physics, and financial mathematics since it can be simply solved with a short time in conversion into the solutions. Also, Classical Black-Scholes equation was extended for arbitrage possibilities [3]. The Black-Scholes equation with arbitrage can be interpreted using quantum mechanic's view point in a sense of an imaginary time from Schrodinger equation of a free particle. Therefore, the Black-Scholes equation including arbitrage possibilities was proposed. Although the Black-Scholes-Schrodinger equation can be used to describe the analysis of option pricing in financial markets, this equation cannot be completely described in the physical meaning of the actual financial market [3].

The Black-Scholes-Schrodinger equation is a quantum financial model which is used for analyzing fair prices of options in real financial market. This equation interprets the Black- Scholes equation with arbitrage possibilities in quantum mechanic's view point in the senses of the Schrodinger equation. The Black-Scholes equation with arbitrage possibilities in the domain $(S,t)\in\mathbb{R}\times[0,T]$ is presented in the form of Equation (1).

$$\frac{\partial \pi(S,t)}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2\pi(S,t)}{\partial S^2}+\frac{r\sigma-\bar{\alpha}f(t)}{\sigma-f}\left(S\frac{\partial \pi(S,t)}{\partial S}-\pi(S,t)\right)=0, \tag{1.1}$$

where $\pi(S,t)$ is the option price at underlying asset price S with time t, S represents an underlying asset price, t represents time variable, σ represents the volatility of underlying asset price, r is the risk free interest rate, T is the expiration date, and the $f(t)$ is called the arbitrage bubble

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function. In case of $f = 0$, Equation (1.1) is reduced to the original Black- Scholes equation with arbitrage possibilities. By using the changing variable technique in $\xi = \ln S$, we obtain

$$\frac{\partial \pi(e^{\xi},t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \pi(e^{\xi},t)}{\partial \xi^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial \pi(e^{\xi},t)}{\partial \xi} + \frac{(r-\bar{\alpha})f(t)}{\sigma - f}\left(S\frac{\partial \pi(e^{\xi},t)}{\partial \xi} - \pi(e^{\xi},t)\right) - r\pi(e^{\xi},t) = 0. \tag{1.2}$$

By a new variable $x = \xi - (r - \frac{\sigma^2}{2})t$ in [3], and so by definiting $\psi(x,t) = e^{r(T-t)}\pi(x,t)$, the Black-Scholes-Schrodinger equation in the domain $(x,t) \in \mathbb{R} \times [0,T]$ is as follows:

$$\frac{\partial \psi(x,t)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 \psi(x,t)}{\partial x^2} + v(x,t)\left(\frac{\partial \psi(x,t)}{\partial x} - \psi(x,t)\right) = 0, \tag{1.3}$$

where $\psi(x,t)$ represents a wave function at time t, $v(x,t)$ is a potential function, $v(x,t) = (r-\bar{\alpha})\frac{\hat{f}(x,t)}{\sigma - \hat{f}(x,t)}$, $\hat{f}(x,t) = f(e^{x+(r-\frac{\sigma^2}{2})t}, t)$, and x is called a space variable. However, the Black-Scholes-Schrodinger equation is not completely consistent with the actual financial market. Therefore, the fractional calculus is used to apply in the Black-Scholes-Schrodinger equation to describe occurrences in financial market especially in field of log-price probability and to specify the variability in prices. The fractional Black-Scholes-Schrodinger equation in the domain $(x,t) \in \mathbb{R} \times [0,T]$ can be expressed as [3]

$$\mathcal{D}_t^{\alpha}\psi(x,t) + \frac{1}{2}\sigma^2 \frac{\partial^2 \psi(x,t)}{\partial x^2} + v(x,t)\left(\frac{\partial \psi(x,t)}{\partial x} - \psi(x,t)\right) = 0, \tag{1.4}$$

where the fractional Caputo's derivative is [5]

$$\mathcal{D}_t^{\alpha}\psi(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial \psi(x,\tau)}{\partial \tau}(t-\tau)^{-\alpha}d\tau, \quad 0 < \alpha \le 1.$$

Equation (1.4) is reduced to the original Black-Scholes-Schrodinger equation when $\alpha = 1$.

2. Main result

For positive integer number N, let $h_t = \frac{T}{N}$ denotes the step size of time variable t. So we define $t_k = kh_t, \quad k = 0,1,\ldots,N$. Using Lemma 2 from [1], for $g(t) \in C^2[0,t_k]$ and $0 < \alpha < 1$ we have

$$\mathcal{D}_t^{\alpha} g(t_k) = \frac{1}{h_t^{\alpha}\Gamma(2-\alpha)}\left(b_0 g(t_k) - \sum_{m=1}^{k} -1(b_{k-m-1} - b_{k-m})g(t_m) - b_{k-1}g(t_0)\right) + \mathcal{O}(h_t^{2-\alpha}), \tag{2.1}$$

where $b_m = (m+1)^{1-\alpha} - m^{1-\alpha}$. The Gegenbauer polynomial of order n associated with the real parameter λ, $\lambda > -\frac{1}{2}, \lambda \neq 0$, denoted by $C_n^{(\lambda)}(x)$ in $x \in [-1,1]$, is given in [2] through the recurrence formula
$C_{n+1}^{(\lambda)}(x) = \frac{2(\lambda+n)}{n+1}xC_n^{(\lambda)}(x) - \frac{2\lambda+n-1}{n+1}C_{n-1}^{(\lambda)}(x), \quad n \geq 1$, with $C_0^{(\lambda)}(x) = 1, C_1^{(\lambda)}(x) = 2\lambda x$, as the first terms. We consider Black-Scholes-Schrodinger equation (1.4) in spatial domain $x \in [-1,1]$, and define $U_k(x) = \psi(x,t_k)$ of the form $U_k(x) = \sum_{j=1}^{M} c_{jk}C_j^{(\lambda)}(x)$ as the spectral solution of Black-Scholes-Schrodinger equation (1.4) in time t_k. By using of relation $C_n^{\lambda}(x) = \frac{1}{2(n+\lambda)}(\partial_x C_{n+1}^{\lambda}(x) - \partial_x C_{n-1}^{\lambda}(x))$, for $\Phi_N^{(\lambda)}(x) = [C_0^{(\lambda)}(x) \quad C_1^{(\lambda)}(x) \quad \cdots \quad C_N^{(\lambda)}(x)]^T$ we have

$$\partial_x(\Phi_N^{(\lambda)}(x)) = D_{(\lambda)}\Phi_N^{(\lambda)}(x),$$

where for even N

$$D_{(\lambda)} = 2\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda+1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda & 0 & \lambda+2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda+1 & 0 & \lambda+3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda & 0 & \lambda+2 & 0 & \lambda+4 & \cdots & \lambda+N-2 & 0 & 0 \\ 0 & \lambda+1 & 0 & \lambda+3 & 0 & \cdots & 0 & \lambda+N-1 & 0 \end{bmatrix}.$$

and for odd N

$$D_{(\lambda)} = 2\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda+1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda & 0 & \lambda+2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda+1 & 0 & \lambda+3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \lambda+1 & 0 & \lambda+3 & 0 & \cdots & \lambda+N-2 & 0 & 0 \\ \lambda & 0 & \lambda+2 & 0 & \lambda+4 & \cdots 0 & 0 & \lambda+N-1 & 0 \end{bmatrix}.$$

Now, we summarize the above formulae as the following

$$D_{(\lambda)} = (d_{ij}) = \begin{cases} 2(\lambda+j), & i=0,1,\ldots,N, \quad \text{for } j=i-k, \begin{cases} k=1,3,5,\ldots,N & \text{if } N \text{ odd} \\ k=1,3,5,\cdots,N-1 & \text{if } N \text{ even} \end{cases} \\ 0, & \text{otherwise.} \end{cases}$$

Here, we consider Gegenbauer-Gauss-Lobatto(GGL) nodes as the real simple roots $x_i \in [-1,1]$ of $(1-x^2)C_{M-1}^{(\lambda)}(x)$ for $i=0,1,\ldots,M$. We consider Black-Scholes-Schrodinger equation (1.4) in points (x_i,t_k), we can write

$$\frac{1}{h_t^\alpha \Gamma(2-\alpha)}\left(b_0\mathbf{c}_k\Phi_M^{(\lambda)}(x_i) - \sum_{m=1}^{k}(b_{k-m-1}-b_{k-m})\mathbf{c}_m\Phi_M^{(\lambda)}(x_i) - b_{k-1}\mathbf{c}_0\Phi_M^{(\lambda)}(x_i)\right)$$
$$+\frac{\sigma^2}{2}\mathbf{c}_k D_{(\lambda)}^2\Phi_M^{(\lambda)}(x_i) + v_{ik}(\mathbf{c}_k D_{(\lambda)}\Phi_M^{(\lambda)}(x_i) - \mathbf{c}_k\Phi_M^{(\lambda)}(x_i)) = 0,$$

where $\mathbf{c}_k = [c_{0k}, c_{1k}, c_{2k}, \ldots, c_{M,k}]$ and $v_{ik} = v(x_i,t_k)$. If we consider initial condition $\psi(x,0) = g(x)$ and boundary condition $\psi(-1,t) = u_1(t)$ and $\psi(1,t) = u_2(t)$, so we have $g(x_i) = \mathbf{c}_0\Phi_M^{(\lambda)}(x_i)$, $u_1(t_k) = \mathbf{c}_k\Phi_M^{(\lambda)}(-1)$ and $u_2(t_k) = \mathbf{c}_k\Phi_M^{(\lambda)}(1)$, respectively. Consequently, by solving the following matrix equation we obtain the unknown coefficient c_{ij} in matrix $X = c_{ij}$,

$$XA = B, \tag{2.2}$$

where the coefficient matrices A and B is obtained from the above linear equations, beside each others. For solving the matrix equation (2.2), the restarted global GMRES (GlGMRES(k)) Algorithm 2 proposed in [4] can be used. We exploit Algorithm 1 to construct an F-orthonormal basis $V_1, V_2, \ldots, V_k$ for the corresponding matrix Krylov subspace $\mathcal{K}_k(A, V_1)$, associated with the matrix equation (2.2). In Algorithm 1 $\|A\|_F$ is the Ferobenius norm of the matrix A, defined by $\|A\|_F = \sqrt{\mathrm{tr}(A^T A)}$. As mentioned in [4], to save memory and CPU-time requirements, the global GMRES method should be utilized in a restarted mode.

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Algorithm 1 Modified global Arnoldi algorithm

1: Choose a matrix V_1 such that $\|V_1\|_F = 1$ $(V_1 = V/\|V\|_F)$
2: For $j = 1, 2, \ldots, k$ Do
: $\widetilde{V} = V_j A.$
: For $i = 1, 2, \ldots, j$ Do
: $h_{i,j} = \langle \widetilde{V}, V_i \rangle_F,$
: $\widetilde{V} = \widetilde{V} - h_{i,j} V_i,$
: End Do
: $h_{j+1,j} = \|\widetilde{V}\|_F$. If $h_{j+1,j} = 0$ then Stop
: $V_{j+1} = \widetilde{V}/h_{j+1,j}$
: End Do

Algorithm 2 Global GMRES(k) algorithm

1: Choose X_0, a tolerance ϵ and set $iter = 0$.
Compute $R_0 = B - X_0 A$, $\beta = \|R_0\|_F$, and $V_1 = R_0/\beta$
2: Construct the F-orthonormal basis $V_1, V_2, \ldots, V_k$ by Algorithm 1
3: Determine y_k as solution of the least square problem:

$$\min_{y \in \mathbb{R}^k} \|\beta e_1 - \widetilde{H}_k y\|_2$$

4: Compute $X_k = X_0 + \mathcal{V}_k (y_k \otimes \mathbf{I})$
5: Compute the residual R_k and $\|R_k\|_F$
6: If $\|R_k\|_F < \epsilon \|R_0\|_F$ Then Stop
7: else $X_0 = X_k$, $R_0 = R_k$, $\beta = \|R_0\|_F$, $V_1 = R_0/\beta$ and $iter = iter + 1$, Go to 2

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