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On the variable parameter Uzawa method for double saddle point systems

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Abstract. In this paper, we present variable parameter Uzawa method for solving double saddle point systems. We find the variable parameters of the proposed method, in a way that minimize some vector norms induced by symmetric positive definite matrices. Some numerical results are given to demonstrate the efficiency of the presented method.

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1. Introduction

In this work, we consider the following large and sparse system of linear equations

$$\mathcal{A}\mathbf{u} \equiv \begin{pmatrix} A & B^T & C^T \\ B & 0 & 0 \\ C & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \equiv \mathbf{b}, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite(SPD) matrix, $B \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{p\times n}$ have full row ranks, $x, b_1 \in \mathbb{R}^n$, $y, b_2 \in \mathbb{R}^m$ and $z, b_3 \in \mathbb{R}^p$. This is a class of double saddle point problems. The notation $\mathrm{Ran}(A)$ stands for the range of A. For given vectors x, y and z of dimension n, m and p, respectively, $\mathbf{u} = (x; y; z)$ will denote a column vector of dimension $n + m + p$. we use $\langle\cdot,\cdot\rangle$ for usual inner product of two vectors. For a symmetric positive definite matrix G, we consider $\|x\|_G = \|G^{\frac{1}{2}}x\|_2$ for an arbitrary vector x, where $\|v\|_2 = \sqrt{\langle v, v\rangle}$ is Euclidean vector 2-norm. Linear systems of the form (1.1) arise from mixed finite element approximation of the potential fluid flow problems; see [1, 2] and the references therein for detailed descriptions of these problems. The following Proposition given in [1] represents the necessary and sufficient condition of the invertibility of the coefficient matrix $\mathcal{A}$ in (1.1).

Proposition 1.1. *Let A be a SPD matrix and assume that B and C have full column ranks. Then a necessary and sufficient condition for the invertibility of the matrix $\mathcal{A}$ in (1.1) is that* $\mathrm{Ran}(B^T) \cap \mathrm{Ran}(C^T) = \{0\}$.

2. Variable parameter Uzawa method

Uzawa's method has long been a popular technique for solving saddle point problems. We study possible extension of Uzawa's method to the double saddle point problem (1.1). To this end, we first split the coefficient matrix $\mathcal{A}$ as follows

$$\mathcal{A} = \mathcal{M} - \mathcal{N}, \quad \mathcal{M} = \begin{pmatrix} A & 0 & 0 \\ B & -\alpha Q & 0 \\ C & 0 & -\beta M \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} A & -B^T & -C^T \\ 0 & -\alpha Q & 0 \\ 0 & 0 & -\beta N \end{pmatrix}, \tag{2.1}$$

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in which the parameters $\alpha > 0$ and $\beta > 0$ and the matrices Q and $M = N$ are given and be positive definite. So, based on the splitting (2.1) the exact solution $\mathbf{u}^* = (x^*; y^*; z^*)$ satisfies

$$x^* = A^{-1}(b_1 - B^T y^* - C^T z^*), \tag{2.2}$$

$$y^* = y^* + \alpha Q^{-1}(Bx^* - b_2), \tag{2.3}$$

$$z^* = z^* + \beta M^{-1}(Cx^* - b_3). \tag{2.4}$$

By applying the first-order Richardson iterative method to the three linear equations (2.2)-(2.4), it follows

$$\begin{aligned} x^{(k+1)} &= A^{-1}(b_1 - B^T y^{(k)} - C^T z^{(k)}), \\ y^{(k+1)} &= y^{(k)} + \alpha Q^{-1}(Bx^{(k+1)} - b_2), \\ z^{(k+1)} &= z^{(k)} + \beta M^{-1}(Cx^{(k+1)} - b_3). \end{aligned} \tag{2.5}$$

We find the optimum parameters α and β such that the norms

$$\|\alpha Q^{-1} g^{(k)} - g^{(k)}\|_Q = \|\alpha Q^{-\frac{1}{2}} g^{(k)} - Q^{\frac{1}{2}} g^{(k)}\|_2, \quad \|\beta M^{-1} h^{(k)} - h^{(k)}\|_M = \|\beta M^{-\frac{1}{2}} h^{(k)} - M^{\frac{1}{2}} h^{(k)}\|_2$$

are minimized, respectively. Here $g^{(k)} = Bx^{k+1} - b_2$ and $h^{(k)} = Cx^{k+1} - b_3$. A direct calculation gives

$$\alpha = \frac{\langle Q^{-1} g^{(k)}, g^{(k)} \rangle}{\langle g^{(k)}, g^{(k)} \rangle},$$

$$\beta = \frac{\langle M^{-1} h^{(k)}, h^{(k)} \rangle}{\langle h^{(k)}, h^{(k)} \rangle}.$$

We are now ready to formulate the variable parameter Uzawa(VPU) method by (2.5) and motivated from [4] for the double saddle point problem (1.1).

Algorithm 1. (Variable parameter Uzawa method)

Given $x^{(0)} \in \mathbb{R}^n$, $y^{(0)} \in \mathbb{R}^m$ and $z^{(0)} \in \mathbb{R}^p$, the sequence $\mathbf{u}^{(k)} = (x(k); y^{(k)}; z^{(k)})$ is defined for $k = 1, 2, \ldots$ as follows:

1. Set $x^{(k+1)} = A^{-1}(b_1 - B^T y^{(k)} - C^T z^{(k)})$.

2. Compute $g^{(k)} = Bx^{k+1} - b_2$ and $d^{(k)} = Q^{-1} g^{(k)}$. Then, compute the relaxation parameter

$$\alpha_k = \begin{cases} \frac{\langle d^{(k)}, g^{(k)} \rangle}{\langle g^{(k)}, g^{(k)} \rangle}, & g^{(k)} \neq 0, \\ 1, & g^{(k)} = 0. \end{cases}$$

   Set $y^{(k+1)} = y^{(k)} + \alpha_k d^{(k)}$.

3. Compute $h^{(k)} = Cx^{k+1} - b_3$ and $s^{(k)} = M^{-1} h^{(k)}$. Then, compute the relaxation parameter

$$\beta_k = \begin{cases} \frac{\langle s^{(k)}, h^{(k)} \rangle}{\langle h^{(k)}, h^{(k)} \rangle}, & h^{(k)} \neq 0, \\ 1, & h^{(k)} = 0. \end{cases}$$

   Set $z^{(k+1)} = z^{(k)} + \beta_k s^{(k)}$.

Remark 2.1. To further improvement of the computing efficiency of the VPU method, we can employ the Cholesky decomposition to solve the systems of linear equations with coefficient matrices A, Q and M, directly. For iterative scheme, one can use conjugate gradient method to some prescribed accuracy at each step.

3. Numerical experiments

We now describe some numerical experiments were carried out in order to show the efficiency and accuracy of the presented method. The computational study was done in the next problems.

Example 3.1. Let us consider the double saddle point system (1.1), where the entries of the matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{m\times n}$, and $C \in \mathbb{R}^{p\times n}$ are defined as follows

$$A = (a_{ij}) = \begin{cases} i+1, & i=j \\ 1, & |i-j|=1 \\ 0, & otherwise. \end{cases} \quad , \quad B = (b_{ij}) = \begin{cases} i, & j = n-m+i \\ 0, & otherwise. \end{cases}$$

and

$$C = (c_{ij}) = \begin{cases} i, & i=j \\ 0, & otherwise. \end{cases}$$

For this problem, the condition of Proposition 1.1 is satisfied. First, we set $Q = BA^{-1}B^T$ and $M = CA^{-1}C^T$, then we used Algorithm 1. to solve (1.1). The vector $\mathbf{b}$ is chosen so that the components of the exact solution $\mathbf{u}$ of (1.1) have values equal to 1. All runs are started with the initial zero vector and terminated if the current iterations satisfy $ERR = \frac{\|\mathbf{r}^{(k)}\|_2}{\|\mathbf{r}^{(0)}\|_2} \leq 10^{-4}$, or if the prescribed iteration number $k_{\max} = 2000$ is exceeded. Here, we define $\mathbf{r}^{(k)}$ as

$$\mathbf{r}^{(k)} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} A & B^T & C^T \\ B & 0 & 0 \\ C & 0 & 0 \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix} .$$

We compare the performance of our method with the SOR-like method [3] by reporting the number of iterations(minIT), the CPU time and the relative residual norm(ERR) in Talbe 1. we choose $\omega = 1.2538$ in the SOR-like method.

Table 1: CPU time, iteration number and EER

			VPU method			SOR-like method			
n	m	p	minIT	CPU(s)	ERR	ω	minIT	CPU(s)	ERR
50	30	10	79	0.0042	9.891e-05	1.2538	285	0.0625	9.996e-05
80	40	20	86	0.0054	9.766e-05	1.2538	429	0.2188	9.978e-05
100	50	40	183	0.0138	9.601e-05	1.2538	530	0.3281	9.914e-05
300	150	80	359	0.1195	9.920e-05	1.2538	1573	4.4688	9.983e-05

From the results reported in Table 1, we can conclude that minIT and computational CPU time are important items to demonstrate the efficiency of the VPU method in comparison with the SOR-like method [3].

References

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